Random Matrix Theory Inspired Passive Bistatic Radar Detection of Low-Rank Signals

Sandeep Gogineni[1]*, Pawan Setlur[1], Muralidhar Rangaswamy[2], Raj Rao Nadakuditi[3]
[1] Wright State Research Institute, OH, USA
[2] Air Force Research Laboratory, Wright Patterson Air Force Base, OH, USA
[3] Electrical Engineering and Computer Science Department,
University of Michigan, MI, USA

***Abstract*—For passive bistatic radar with a noisy reference signal, we propose a singular value decomposition (SVD) and Eigen detector that significantly outperforms the conventional cross-correlation detector. We consider the scenario when the signals of opportunity across several independent snapshots/pulses span a low-rank signal space. The target reflectivity is assumed to change independently from one pulse to another within a processing interval. We demonstrate this performance improvement through extensive numerical simulations across various surveillance and reference signal-to-noise ratio (SNR) regimes.**

Index Terms: Passive radar, Random matrix theory, Phase transition, Singular value decomposition, Kolmogorov-Smirnov, Detection

I. Introduction

Passive radar systems have access to a noisy replica of the transmitted signal, obtained through a dedicated reference channel. Historically, passive radar detection has been performed by computing a test statistic that is a cross correlation between received target surveillance signal and this noisy reference signal. However, no claims can be made with regards to the optimality of such a detector [1]. Recently, the authors of [2] have studied this problem and derived the GLRT detector for passive multistatic radar. The model proposed in [2] has two major limitations. Firstly, it considers the entire transmitted waveform to be *deterministic unknown* without any structure. In practice, this is never the case, as signals from terrestrial or space based communication systems exhibit low-rank structure. Secondly, the target model in [2] assumes that the target reflectivity is constant over the entire processing interval. This assumption is flawed. In another recent paper [3], apart from ignoring the underlying low-rank nature, authors considered a random model for the transmitted signal but never consider the scintillating targets in the processing interval.

Overcoming these limitations, in [1], we proposed a novel passive radar detector that exploits the inherent structure in the transmitted signal and also the target reflectivity is assumed to change independently from one pulse to another within a processing interval. However, in [1], we restricted the analysis to rank 1 signals that are typical for digital systems employing amplitude modulation. In this paper, we will consider the scenario when the signals of opportunity across several snapshots/pulses span an arbitrary space of rank K. For example, such an analysis is particularly important when the signals are frequency modulated. In this paper, we extend the SVD based detector in [1] to this scenario. Using numerical simulations, we will demonstrate the performance improvement compared to the traditional cross correlation detector.

II. Signal Model

We consider the hypothesis testing problem for the detection of a target at a given range and Doppler. In order to perform this test, the received data is shifted back by the appropriate delay and Doppler to arrive at the following testing problem

$$H_0 : \begin{cases} \boldsymbol{y}_{si} = \boldsymbol{n}_{si}, \\ \boldsymbol{y}_{ri} = \mu_{ri}\boldsymbol{u}_i + \boldsymbol{n}_{ri}, \end{cases} \quad (1)$$

$$H_1 : \begin{cases} \boldsymbol{y}_{si} = \mu_{si}\boldsymbol{u}_i + \boldsymbol{n}_{si}, \\ \boldsymbol{y}_{ri} = \mu_{ri}\boldsymbol{u}_i + \boldsymbol{n}_{ri}, \end{cases} \quad (2)$$

where $i \in \{1, \ldots, N\}$ denotes the pulse index and the subscripts s and r represent the surveillance and reference channels, respectively. We assume the complex attenuation terms to be statistically independent and Gaussian distributed from one pulse to the other. Without loss of generality (w.l.o.g), μ_{si} and μ_{ri} are zero mean complex Gaussian distributed with variance σ_s^2 and σ_r^2, respectively, where we assume the unit norm transmit pulses $\boldsymbol{u}_i$ to contain M samples.

We assume that each $\boldsymbol{u}_i \in \mathcal{U}$, where $\mathcal{U}$ is a set of cardinality K containing all possible unique transmit pulses. We assume each of these K pulses appears randomly in $\boldsymbol{U}$. While the rank 1 model considered in [1] is true for illuminators employing amplitude and phase modulation schemes, for illuminators using other modulations such as frequency modulation, the waveform changes from pulse to pulse and hence the rank K model presented above is relevant. Further, we assume the attenuation varies from pulse to pulse due to target fluctuations and also due to the random message modulations in the transmitted symbols from pulse to pulse. Note that formulating the detection problem in the above manner by distinguishing between the different pulses is possible only when there is

*Corresponding author email: sandeep.gogineni@wright.edu.
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perfect time synchronization. Most commercial illuminators follow mechanisms to help achieve perfect synchronization between the transmitter and receiver.

Now, we stack the measurement vectors corresponding to different pulses $\boldsymbol{Y}_s = [\boldsymbol{y}_{s1}, \cdots, \boldsymbol{y}_{sN}]$ and $\boldsymbol{Y}_r = [\boldsymbol{y}_{r1}, \cdots, \boldsymbol{y}_{rN}]$. Similarly, define $M \times N$ matrix $\boldsymbol{U} = [\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N]$ and $\boldsymbol{\mu}_s = \operatorname{diag}\{\mu_{s1}, \ldots, \mu_{sN}\}$, $\boldsymbol{\mu}_r = \operatorname{diag}\{\mu_{r1}, \ldots, \mu_{rN}\}$. Therefore, we have

$$H_0 : \begin{cases} \boldsymbol{Y}_s = \boldsymbol{N}_s, \\ \boldsymbol{Y}_r = \boldsymbol{U}\boldsymbol{\mu}_r + \boldsymbol{N}_r, \end{cases} \tag{3}$$

$$H_1 : \begin{cases} \boldsymbol{Y}_s = \boldsymbol{U}\boldsymbol{\mu}_s + \boldsymbol{N}_s, \\ \boldsymbol{Y}_r = \boldsymbol{U}\boldsymbol{\mu}_r + \boldsymbol{N}_r. \end{cases} \tag{4}$$

In this problem, we assume additive noise samples to be independent zero mean Gaussian with variance σ^2. The definition of SNR is separate under both the channels

$$\mathrm{SNR}_s = 20 \log \frac{\sigma_s}{\sqrt{M}\sigma} \mathrm{dB},$$

$$\mathrm{SNR}_r = 20 \log \frac{\sigma_r}{\sqrt{M}\sigma} \mathrm{dB}.$$

III. Detectors

The classical and frequently used approach is to form a test statistic by computing the cross correlation between the surveillance and reference channel measurements

$$T_{\mathrm{CC}} = \sum_{i=1}^{N} \left| \boldsymbol{y}_{si}{}^H \boldsymbol{y}_{ri} \right|^2. \tag{5}$$

Unlike an active radar system, no claims can be made about the optimality of the test statistic in (5) due to the noisy nature of the reference measurements. Consequently, the performance of this detector can be improved upon in some instances. In particular, we observe that this detector performs a correlation on the raw data across both the channels for each of the N snapshots. It does not exploit the fact that the unit-norm transmit pulses across several snapshots ($N >> K$) satisfy a rank K structure. In other words, we would like to exploit the common rank K signal structure inherent to this problem.

Let $\boldsymbol{V}_s$ be a $M \times K$ matrix whose K columns denote the dominant left singular vectors of $\boldsymbol{Y}_s$. Similarly $\boldsymbol{V}_r$ contains the K dominant left singular vectors of $\boldsymbol{Y}_r$. Then, we define the SVD-Eigen test statistic as

$$T_{\mathrm{SVD-Eigen}} = \sum_{k=1}^{K} \sum_{j=1}^{K} \left| S_{sk}^2 \boldsymbol{v}_{sk}{}^H \boldsymbol{v}_{rj} \right|^2,$$

where $\boldsymbol{v}_{sk}$, $\boldsymbol{v}_{rj}$, and S_{sk} denote the k^{th} dominant singular vectors and singular value, respectively. We expect the SVD detector to perform better than the cross correlation detector because the left singular vectors act like a joint estimate of the rank K signal space from N snapshots. Hence, we compute this joint estimate from N snapshots before performing the cross correlation operation instead of doing it on the raw data from each snapshot separately. Further, from random matrix theory [4]– [7], SVD of these matrices have some interesting threshold behavior that can be used for asymptotic performance prediction. For performance comparison, we consider the ideal case when all the parameters including the deterministic pulses $\boldsymbol{u}_i$ are known. The clairvoyant detector in this case is

$$T_{\mathrm{CLAIR}} = \sum_{i=1}^{N} \left| \boldsymbol{y}_{si}{}^H \boldsymbol{u}_i \right|^2$$

IV. Statictical Techniques

In this section we describe statistical techniques used to analyze the performance of the detectors in the previous section. The probability distributions of the test statistic are complicated to derive both analytically and numerically. Hence, we use Kolmogorov-Smirnov (KS) test as a measure of separability between the two hypotheses in (1), for the detectors in the previous section. However, no claims can be made with regards to the probabilities of detection and false alarm.

A. Kolmogorov-Smirnov test: Not just a goodness of fitness test

Typically, the two sample KS [8]– [10] tests whether the two samples belong to a particular distribution against the alternative that they belong to different distributions. The two sample KS statistic is expressed as,

$$KS = \sup_x |\hat{F}_1(x) - \hat{G}_1(x)| \tag{6}$$

where $\hat{F}_1(x), \hat{G}_1(x)$ are the empirical (cumulative) distributions corresponding to the unspecified (cumulative) distributions $F(x), G(x)$, respectively. The empirical distributions are defined as

$$\hat{F}_1(x) = \begin{cases} 0 & \text{if } x < T_{(0,1)} \\ \frac{p}{P} & \text{if } T_{(0,p)} \le x < T_{(0,p+1)}, p = 1, 2, \ldots, P-1 \\ 1 & x \ge T_{(0,P)} \end{cases}$$

$$\hat{G}_1(x) = \begin{cases} 0 & \text{if } x < T_{(1,1)} \\ \frac{p}{P} & \text{if } T_{(1,p)} \le x < T_{(1,p+1)}, p = 1, 2, \ldots, P-1 \\ 1 & x \ge T_{(1,P)} \end{cases}$$

wherein $T_{(o,p)}, T_{(1,p)}, p = 1, 2, \ldots, P$ denote the two (ascending) ordered samples under test, and is explained subsequently. From (6), we notice immediately that if the empirical distributions are well discriminated (separated) the test statistic assumes unity, and when the distributions are not well separated the test statistic assumes smaller positive values.

In our problem, for analyzing how the detectors discriminate between the null and alternate in (1), we first generate P measurements of the test statistic $\mathcal{T}$ under the null as in (1), and order them as $T_{(0,1)}, T_{(0,2)}, \ldots, T_{(0,P)}$ and likewise generate P measurements under the alternate hypothesis in (1) and order them as $T_{(1,1)}, T_{(1,2)}, \ldots, T_{(1,P)}$. The test statistic, $\mathcal{T}$ could be from any one of the detectors mentioned in the previous section. Next, we perform the KS test on these two samples as described above to analyze the discriminating (between the hypotheses in (1)) ability of the detectors T_{CC}

and $T_{SVD-Eigen}$. Note that this approach is employed because the probability distributions of T_{CC} and $T_{SVD-Eigen}$ are complicated to derive under H_0 and H_1 in (1), and is not the main crux of this paper. However, computing these probability distributions is a topic of our ongoing research as they are essential in computing the thresholds for target detection.

B. Phase transition thresholds

From random matrix theory [4]– [7], below a critical threshold region, the dominant eigenvalue and eigenvectors are not sufficiently representative of their true counterparts. The objective here is to provide insights on this threshold since our proposed detector in this paper is a function of the eigenvalue and eigenvectors. Consider realizations of a random vector, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ where $\mathbf{x}_i \in {}^M = \alpha_i \mathbf{u} + \mathbf{n}_i$. Assume that α_i is a zero mean, unit variance random variable not necessarily normally distributed but has a finite fourth order moment. The noise vector, $\mathbf{n}_i$ is zero mean, normally distributed with covariance matrix equal to $\sigma^2\mathbf{I}$, and statistically uncorrelated from the other noise vectors, $\mathbf{n}_j, j = 1, 2, \ldots, N, j \neq i$. Denote $\mathbf{v}_1$ and $\hat{\mathbf{v}}_1$ as the true and estimated dominant eigenvectors, derived from the true covariance matrix, $\mathbf{u}\mathbf{u}^H + \sigma^2\mathbf{I}$, and sample covariance matrix, $\frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i\mathbf{x}_i^H$ respectively. Likewise, denote $\hat{\lambda}_1$ as the estimated dominant eigenvalue estimated from the sample covariance matrix. Then we recall the following theorem from random matrix theory [7].

Theorem 1. *In the joint limit $M, N \to \infty$, we have,*

$$\hat{\lambda}_1 = \begin{cases} \sigma^2\left(1 + \sqrt{\frac{M}{N}}\right)^2, & \text{if } \frac{N}{M} < \frac{\sigma^4}{||\mathbf{v}||^4} \\ \left(||\mathbf{v}||^2 + \sigma^2\right)\left(1 + \frac{M}{N}\frac{\sigma^2}{||\mathbf{v}||^2}\right), & \text{if } \frac{N}{M} \geq \frac{\sigma^4}{||\mathbf{v}||^4} \end{cases} \quad (7)$$

$$|\hat{\mathbf{v}}_1^H \mathbf{v}_1|^2 = \begin{cases} 0, & \text{if } \frac{N}{M} < \frac{\sigma^4}{||\mathbf{v}||^4} \\ \frac{(N||\mathbf{v}||^4/M\sigma^4) - 1}{(N||\mathbf{v}||^4/M\sigma^4) + (||\mathbf{v}||^2/\sigma^2)}, & \text{if } \frac{N}{M} \geq \frac{\sigma^4}{||\mathbf{v}||^4} \end{cases} \quad (8)$$

Theorem.1, (7) states that in the joint limit, the estimated eigenvalue is deterministic, and similarly, from (8) the square of the inner products are deterministic. The threshold in both (7),(8) are identical and are given in blue. We note a somewhat surprisingly result from (8). For N/M below this threshold, the estimated eigenvector is completely orthogonal to its true counterpart, which implies that it offers no representative statistical information about the true dominant eigenvector. Although Theorem. 1 is with regard to the dominant eigenvector of the sample covariance matrix, it is readily applicable in a straightforward manner to the dominant left singular vectors of the stacked matrices $\boldsymbol{Y}_r$ and $\boldsymbol{Y}_s$ in equation (3), as well. In Section V, the threshold will be shown in the simulations for comparisons. In the rank-1 case, we notice an excellent agreement with the theory as predicted by (7), (8). Generalizing the result in Theorem.1 to the case of arbitrary rank K is a topic of our ongoing research.

Remark: In a passive radar problem, both the reference and surveillance SNRs must be above the threshold for the corresponding left singular vectors to not be orthogonal to each other.

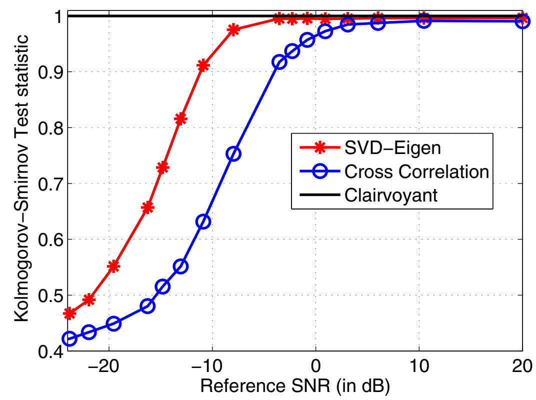

Fig. 1. Kolmogorov-Smirnov Statistic as a function of SNR_r for $\text{SNR}_s = -10$dB, $N = 150$, $M = 11$, and varying pulses ($K = 3$). (Clairvoyant detector does not depend on SNR_r since $\boldsymbol{u}$ is known)

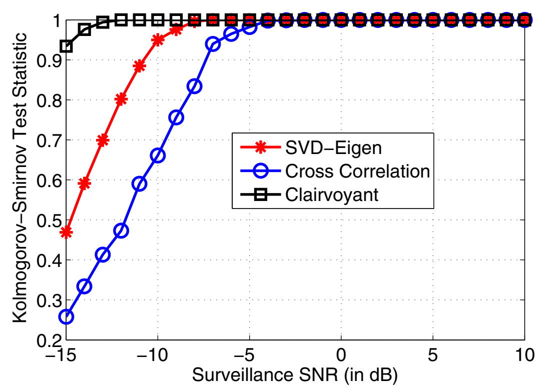

Fig. 2. Kolmogorov-Smirnov Statistic as a function of SNR_s for $\text{SNR}_r = -10$dB, $N = 150$, $M = 11$, and varying pulses ($K = 3$).

V. Numerical Examples

A useful measure to compare the detectors is to employ the two sample Kolmogorov-Smirnov (KS) test by generating samples from both the hypotheses for varying values of N, SNR_s, and SNR_r. The KS test statistic has values varying from 0 to 1 with higher values implying greater separability between the distributions under the null and alternative hypotheses. Therefore, the larger the KS test statistic value the better. In Figs. 1 and 2, we plot the KS test statistic for the case when the rank $K = 3$, $N = 150$, $M = 11$. We clearly observe significant improvement in performance for the SVD-Eigen detector over the CC detector. For comparison, we have also plotted the ideal Clairvoyant detector performance in the same figures. Note that the Clairvoyant detector performance is constant in Fig. 1 because the performance of this detector does not depend on the reference SNR since it assumes perfect true knowledge of the transmitted pulses.

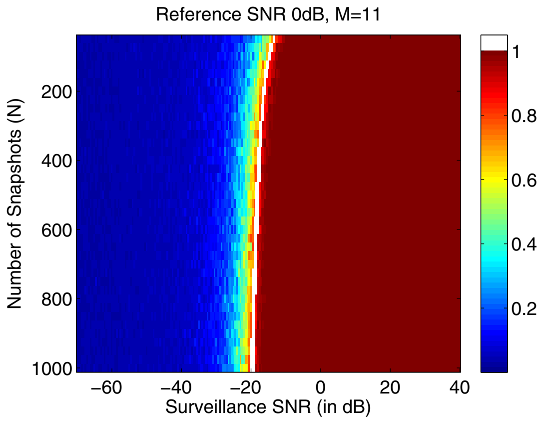

Fig. 3. Kolmogorov-Smirnov Statistic of the SVD-Eigen detector for $\mathrm{SNR}_r = 0\mathrm{dB}$, $M = 11$, $K = 1$.

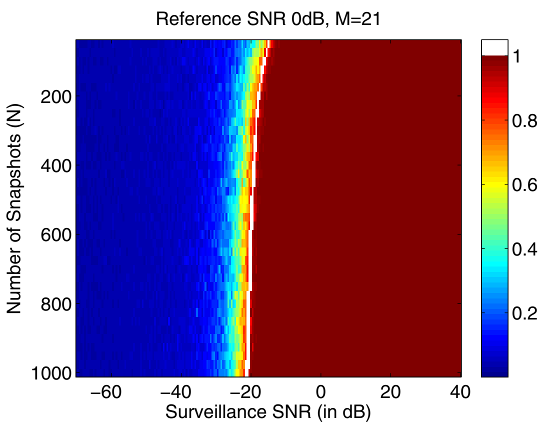

Fig. 4. Kolmogorov-Smirnov Statistic of the SVD-Eigen detector for $\mathrm{SNR}_r = 0\mathrm{dB}$, $M = 21$, $K = 1$.

In Figs. 3 and 4, we plot the KS test statistic across a wide range of values for the number of snapshots and surveillance SNR for a fixed reference SNR. We also plot the theoretical phase transition thresholds computed from equation (8) in white color. We observe from Fig. 3 and 4 that even for relatively small value of $M = 11, 21$, the asymptotic thresholds for SNR_s computed using (8) very accurately capture the performance transition at $\mathrm{SNR}_r = 0\mathrm{dB}$ which is above the phase transition threshold computed for the reference signal. Note that this simulation was performed for the case when $K = 1$.

In Fig. 5, we choose a reference SNR that falls below this threshold. In this scenario, the left singular vector computed from the reference channel measurements does not carry any useful information about the transmitted pulses. Therefore, an inner product of the singular vectors in both the channels cannot be used to distinguish between the hypotheses. However, we still see the distinguishable red region in Fig. 5. The reason for this behavior is that even though SNR_r is below the threshold, the singular value computed from the surveillance channel by itself helps to distinguish between the null and the alternative hypotheses as an energy detector.

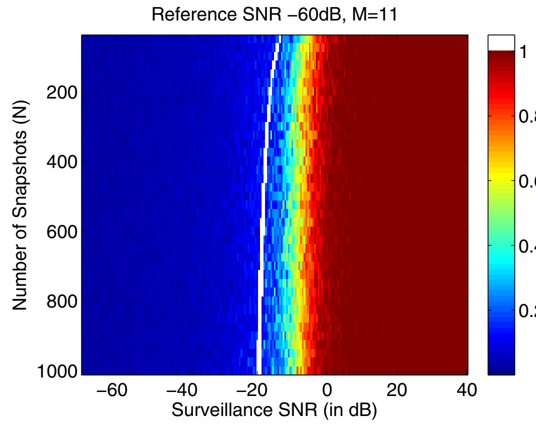

Fig. 5. Kolmogorov-Smirnov Statistic of the SVD-Eigen detector for $\mathrm{SNR}_r = -60\mathrm{dB}$, $M = 11$, $K = 1$.

VI. Concluding Remarks

In this paper, we have proposed an SVD-based detector for illuminators whose signals across multiple snapshots span a low-rank space. We have demonstrated that our proposed detector significantly outperforms the classical cross-correlation detector used in passive radar systems.

In future work, we will analytically derive the probabilities of detection and false alarm for this detector. We will also derive the GLRT detector for this hypothesis testing problem and compare with the SVD-Eigen detector presented in this paper. Further, we will expand our results to the case of multiple illuminators and receivers.

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