COMPUTING THE LARGEST EIGENVALUE DISTRIBUTION FOR NON-CENTRAL WISHART MATRICES

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***Abstract*—Eigenvalues of the Gram matrix formed from received data frequently appear in sufficient detection statistics for multi-channel detection with Generalized Likelihood Ratio (GLRT) and Bayesian tests. In a frequently presented model for passive radar, in which the null hypothesis is that the channels contain only complex white Gaussian noise and the alternative hypothesis is that the channels contain a common rank-one signal in the mean, the GLRT statistic is the largest eigenvalue λ_1 of the Gram matrix formed from data, which has a Wishart distribution. Although exact expressions for the distribution of λ_1 are known under both hypotheses, numerically calculating values of these distribution functions presents difficulties in cases where the dimension of the data vectors is large. Following on recent work addressing this issue under the null hypothesis, this paper presents a method to calculate values of this distribution under the alternative hypothesis, allowing tractable computation of receiver operating characteristic curves.**

***Index Terms*—Wishart matrix, Multi-channel detection, Passive radar, CFAR thresholds**

I. Introduction

Recent research directions in multi-sensor statistical signal processing (e.g., [1]–[6]) and MIMO communications (e.g., [7]–[10]) have brought significant attention to the roles of complex Wishart matrices in these application areas. Wishart matrices have a long history in the statistical literature [11]–[13]; they arise naturally in multi-channel sensing and MIMO applications when the received data is modeled as being complex normally distributed. In particular, statistics used for detection, estimation, and characterization of collected data are often functions of the Gram matrix formed from the received data, which is a complex Wishart matrix under typical Gaussian data models.

This paper examines a problem motivated by multi-channel detection, as arises in multistatic passive radar, where it is to be ascertained whether a common signal is present across several noisy channels. In some such problems, a subset of the channels may be "reference channels" known to contain a noisy copy of the common signal of interest. Or all channels may be "surveillance channels" which may or may not contain the common signal. The surveillance-only scenario is the primary motivation for the analysis in this paper, and is discussed in [14], [15], though the results apply to a broader class of detection problems [16], [17].

The largest eigenvalue of the $M \times M$ Gram matrix formed from the complex data was shown in [18] to be a sufficient statistic for the Generalized Likelihood Ratio Test (GLRT) of a rank-one signal in M channels of zero-mean white Gaussian noise (ZMWGN). Typical multi-receiver detection scenarios involve a relatively small number of sensors (generally $M < 10$), but detection of weak signals may require the length N of data sequences collected at each sensor to be on the order of $10^5 - 10^6$ or larger. To set detection thresholds corresponding to desired false alarm probabilities in such situations, it is thus necessary to compute the distribution of λ_1 under the null hypothesis for small values of M and very large values on N; complete performance analysis of the detectors requires computation of the distribution under the alternative hypothesis as well.

At this point, it is important to emphasize that the distribution of λ_1 is known from classical statistical results in the null hypothesis case [12] and from more recent work in MIMO communications under the alternative hypothesis [19]. The issue addressed in this paper is that these formulations of the distributions are not amenable to numerical evaluation except for relatively small values of M and N. Commonly-used approximation methods, such as those involving the Tracy-Widom distribution, lack sufficient fidelity for the desired ranges on M and N. Monte Carlo methods, including those that incorporate variance reduction methods such as importance sampling [20], [21], have been used to analyze these distributions. But they are not computationally viable in the low false-alarm regimes entailed in radar surveillance applications.

Previous work has made possible the computation of the distribution of λ_1 under the null hypothesis for the large values of N of interest in multi-channel detection problems [22]. This paper extends these techniques to the alternative hypothesis in which a common signal is present in the mean across the channels, in which case the Wishart distribution is referred to as "non-central." This allows for exact tractable analysis of detector performance, including computation of receiver operating characteristic (ROC) curves into the low false-alarm regime. Section II presents the physical model of the passive radar scenario that motivates this work and gives rise to the model for the data. Section III discusses classical and modern results on the distributions of the largest eigenvalue of Wishart matrices and their intractability for this class of detection problem. Techniques similar to those used in [22] are used in section IV to expand the non-central distribution as inner products of generalized Laguerre polynomials. Results from numerical experiments are presented in section V. These techniques allow the distribution of the largest eigenvalue of a non-central complex Wishart matrix to be computed for a very large number of samples, as encountered in multi-channel passive radar detection with weak illuminating signals. This allows for direct computation of ROC curves: previously the null hypothesis distribution could be used to compute detection thresholds corresponding precisely to false-alarm probabilities, while detection probabilities for these thresholds could only be analyzed via Monte Carlo simulation.

II. Signal Model

Consider the multi-static passive radar scenario presented in [15]. There are assumed to be M receivers and a single emitter/scatterer in known positions, and the presence or absence of a target with a hypothesized position and velocity state is to be ascertained. It is assumed that there is not a direct-path receiver; i.e., the emitted signal only manifests by scattering off the target. It is assumed there is no clutter.

The signal at each receiver is assumed under both the signal absent $\mathcal{H}_0$ and signal present $\mathcal{H}_1$ hypotheses to contain ZMWGN. Under the $\mathcal{H}_1$ hypothesis, it is assumed there is a common rank-one signal with each channel imparting a scalar gain, time delay, and Doppler to the transmitted signal. The data at each receiver $\tilde{x}_m$ is then corrected for the posited physical state x_m. Thus, under the null hypothesis $\mathcal{H}_0$ that the received data contain only noise, these vectors x_m, $m = 1, \ldots, M$ are given by

$$\mathcal{H}_0 : x_m = \xi_m$$

where the ξ_m are independent N-vectors of zero-mean complex Gaussian noise. Under the alternative hypothesis,

$$\mathcal{H}_1 : x_m = \eta_m s + \xi_m$$

where η_m is a complex channel gain and s is a complex N-vector representing the common signal component across all M channels. These vectors can then be used to construct the $N \times M$ matrix $X = [x_1, \ldots, x_m]$. Let $U = [\eta_1 s, \ldots, \eta_m s]$ be the mean matrix containing the signal under $\mathcal{H}_1$, and let μ_1 be the single non-zero eigenvalue of $U^\dagger U$. The GLRT statistic for this detection problem when s is rank-one has been shown to be the largest eigenvalue λ_1 of the Gram matrix $X^\dagger X$ [18].

III. Numerical Limitations of the CDF

A lemma given in [23] on the general structure of the CDF of λ_1 for the central uncorrelated, central correlated, and non-central uncorrelated cases of complex Wishart matrices implies that the CDF of λ_1 in each case may be written as the determinant of some matrix. Using the lemma, the CDF of λ_1 under $\mathcal{H}_0$ is equivalent to the form originally derived in [12], and under $\mathcal{H}_1$ it can be written as

$$F_{\lambda_1}(x) = \frac{1}{C} \left| \begin{cases} \int_0^x {}_0F_1(N-M+1, \mu_1 t) t^{N-i} e^{-t} dt, \\ \qquad j = 1, i = 1, \ldots, M \\ \int_0^x t^{N+M-i-j} e^{-t} dt, \\ \qquad j = 2, \ldots, M, i = 1, \ldots, M \end{cases} \right| \quad (1)$$

In this expression, C is a normalizing constant equal to the limit of the determinant term as $x \to \infty$.

Note that, for columns $j = 2, \ldots, M$, the elements take the form of a lower incomplete gamma function, defined as

$$\gamma(N+M-i-j+1, x) = \int_0^x t^{N+M-i-j} e^{-t} dt$$

In principle, equation (1) can be used to compute probabilities, but in practice the extremely large gamma function values lead to overflow of double precision floating point arithmetic for values of N much smaller than those of interest in radar detection applications. This is shown in Table I.

TABLE I
Computational limits of expression from [23] in double precision floating point arithmetic

Sensors (M)	Maximum Samples (N)
2	101
3	75
4	57
5	47

In a practical problem, N on the order of $10^5 - 10^6$ or greater would be required. Although various methods to improve upon this problem for the null hypothesis (central) distribution have been explored [15], [22], to the best of the authors' knowledge no such work exists for the alternative hypothesis (non-central) distribution.

IV. Laguerre Conjugation and a Change of Variables

Let $\Xi(x)$ be the $M \times M$ matrix given in the determinant form of F_{λ_1} in equation (1), and let A be the lower triangular matrix of normalized (in the Laguerre inner product sense) coefficients for the generalized Laguerre polynomials $L_i^{(a)}$ where the parameter $a = N - M$. Under the $\mathcal{H}_1$ hypothesis, $M-1$ columns of $\Xi(x)$ contain incomplete gamma functions. Thus, similar to the central case, conjugating $\Xi(x)$ by A followed by well chosen variable changes will allow cancellation of the extremely large intermediate terms generated by the gamma and hypergeometric functions in the matrix entries.

Consider the conjugation of $\Xi(x)$ by A, denoted $\Psi(x) = A^\dagger \Xi(x) A$. An arbitrary element of the matrix $\Psi_{ij}(x)$ is

$$\Psi_{ij}(x) = \sum_{l=i}^{M} A^\dagger_{il} \sum_{k=j}^{M} \Xi_{lk}(x) A_{kj}$$

Noting that, due to the lower triangular structure of A, any terms in the summation where $l < i$ and $k < j$ are zero. This expansion may be divided into two cases: the first column containing hypergeometric function terms, and the other $M-1$ columns containing only incomplete gamma functions. First, consider the case that $j = 1$. Substituting the elements of $\Xi(x)$ and the corresponding generalized Laguerre coefficients constituting the elements of A into the above summation results in

$$\Psi_{ij}(x) = \sqrt{\frac{i!j!}{(a+i)!(a+j)!}} \sum_{l=i}^{M} \Bigg(L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-1} \int_0^x e^{-t} {}_0F_1(N-M+1, \mu_1 t) t^{N-l} dt + \sum_{k=2}^{M} L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-k} \int_0^x e^{-t} t^{N+M-l-k} \Bigg) dt$$

Note that $L^{(a)}_{i,j}$ denotes the coefficient corresponding to the t^j term of the generalized Laguerre polynomial $L^{(a)}_i$. Next, rearrange the sums and integrals, pulling the factor of $e^{-t}t^{N-M} = e^{-t}t^a$ out from each term in the summations. This yields

$$\Psi_{ij}(x) = \sqrt{\frac{i!j!}{(a+i)!(a+j)!}} \int_0^x e^{-t} t^a \times \sum_{l=i}^{M} \Bigg(L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-1} {}_0F_1(N-M+1, \mu_1 t) t^{M-l} + \sum_{k=2}^{M} L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-k} t^{2M-l-k} \Bigg) dt$$

In this form, it is clear that to make computation of $\Psi_{ij}(x)$ numerically tractable, the leading integrand term arising from the incomplete gamma functions and the leading constants on the order of $a!$ must be eliminated. This is accomplished through a change of variables, substituting $t \to a + u\sqrt{2a}$. Making this substitution, the elements become

$$\Psi_{ij}(x) = \sqrt{\frac{i!j!}{(a+i)!(a+j)!}} \times \int_{-\sqrt{a/2}}^{y(x)} e^{-(a+u\sqrt{2a})} (a+u\sqrt{2a})^a \times \sum_{l=i}^{M} L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-1} \times {}_0F_1(N-M+1, \mu_1(a+u\sqrt{2a}))(a+u\sqrt{2a})^{M-l} + \sum_{k=2}^{M} L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-k} (a+u\sqrt{2a})^{2M-l-k} \sqrt{2a}\, du$$

where $y(x) = \frac{x-a}{\sqrt{2a}}$. Define the following functions:

$$c_{ij}(a) = \frac{e^{-\epsilon(a)}}{\sqrt{\prod_{k=1}^{i}(1+k/a) \prod_{k=1}^{j}(1+k/a)}}$$

$$\epsilon(a) = \log a! - a \log a + a - \frac{1}{2} \log 2\pi a$$

$$\phi_a(t) = a \log(1 + t\sqrt{2/a}) - t\sqrt{2a}$$

factoring the constant terms out of the integrand and substituting in the above functions, Ψ_{ij} can be written as

$$\Psi_{ij}(x) = \frac{c_{ij}(a) a^{-i/2} a^{-i/2} \sqrt{i!j!}}{\sqrt{\pi}} \int_{-\sqrt{a/2}}^{y(x)} e^{\phi_a(u)} \sum_{l=i}^{M} L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-1} \times {}_0F_1(N-M+1, \mu_1(a+u\sqrt{2a}))(a+u\sqrt{2a})^{M-l} + \sum_{k=2}^{M} L^{(a)}_{M-i,M-l} L^{(a)}_{M-1,M-k} (a+u\sqrt{2a})^{2M-l-k} du$$

Define the family of polynomials

$$D^{(a)}_n(t) = (-1)^n n! (2/a)^{n/2} L^{(a)}_n (a + t\sqrt{2a})$$

Note that a consequence of this definition is the family satisfying the orthogonality relation

$$\int_{-\sqrt{a/2}}^{\infty} D^{(a)}_n(t) D^{(a)}_m(t) e^{\phi_a(t)} dt = \frac{\sqrt{\pi} 2^n n!}{c_{nn}(a)} \delta_{nm}$$

Then substituting the coefficients for the D polynomials for the generalized Laguerre coefficients, Ψ_{ij} becomes

$$\Psi_{ij}(x) = \frac{c_{ij}(a)}{\sqrt{\pi 2^{i+j} i!j!}} \int_{-\sqrt{a/2}}^{y(x)} e^{\phi_a(u)} \sum_{l=i}^{M} D^{(a)}_{M-i,M-l} D^{(a)}_{M-1,M-1} u^{M-l} {}_0F_1(N-M+1, \mu_1(a+u\sqrt{2a})) + \sum_{k=2}^{M} D^{(a)}_{M-i,M-l} D^{(a)}_{M-1,M-k} u^{2M-l-k} du \quad (2)$$

This completes the calculation. Next, consider the elements $\Psi_{ij}(x)$ for the case $j = 2, \ldots, M$. It is clear the derivation will follow as shown in this section, without the hypergeometric function in the leading part of the summation. This allows these elements to be written more concisely as partial inner products of the D polynomials.

$$\Psi_{ij}(x) = \frac{c_{ij}(a)}{\sqrt{\pi 2^{i+j} i!j!}} \int_{-\sqrt{a/2}}^{y(x)} D^{(a)}_i(t) D^{(a)}_j(t) e^{\phi_a(t)} dt \quad (3)$$

Therefore, the cumulative distribution function for λ_1 can be written as the determinant of Ψ; i.e.,

$$F_{\lambda_1}(x) = |\Psi_{ij}(x)| \quad (4)$$

such that the matrix elements $\Psi_{ij}(x)$ are given by equations (2) and (3).

V. Numerical Results

This section demonstrates numerical computation of the distribution of λ_1 in the non-central case. The CDF is computed using the D-polynomial formula shown in equations (2) and (3), and compared with the original gamma and hypergeometric function formula shown in equation (1) for a relatively small problem size. The utility of the D-polynomial formulation for much larger problem sizes is demonstrated via comparison to Monte Carlo simulation data. Subsequently, the non-central formula is used in conjunction with previous results on the distribution of λ_1 in the central case to compute a ROC curve.

A. CDF Comparison

In this section, the method for computing the CDF of λ_1 presented in section IV is compared in small problem sizes with equation (1), and for larger problem sizes against a Monte Carlo simulation. The signal in the mean is taken to be N recorded samples of a QPSK signal. The non-zero eigenvalue of the mean matrix Gramian μ_1 is a function of the per-channel SNR. Consider first the small problem comparing equations (1) and (4).

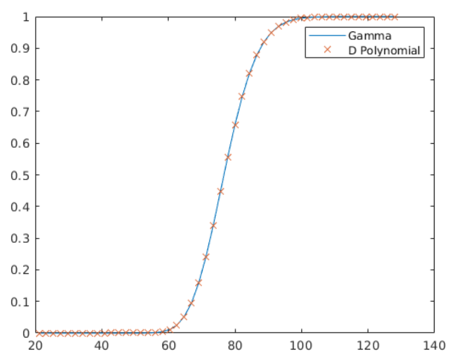

Fig. 1. Comparison of $F_{\lambda_1}(x)$ calculated using equation (1) and equations (2), (3) for $M = 2$ and $N = 64$ with -5 dB per channel SNR

Note the absolute mean-squared error between the two methods is on the order of machine epsilon.

Next, consider a larger problem, which must be addressed with Monte Carlo methods as (1) overflows double precision floating point arithmetic for values of N larger than those shown in Table I.

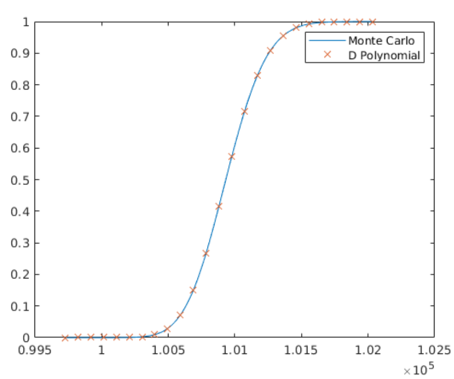

Fig. 2. Comparison of $F_{\lambda_1}(x)$ calculated using equation (4) and a Monte Carlo simulation with 10^6 trials for $M = 4$ and $N = 10^5$ with -15 dB per channel SNR

B. Computing ROC Curves

In a passive radar application, the primary goal of computationally tractable formulas for the GLRT statistic under both the $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses is to compute exactly the probability of false alarm $P_f = 1 - F_{\lambda_1}(T)$ under $\mathcal{H}_0$ and $P_d = 1 - F_{\lambda_1}(T)$ under $\mathcal{H}_1$. The threshold value T is generally set to maintain a constant P_f as required for a particular system's operation. Previously, the $\mathcal{H}_1$ case could only be approached via Monte Carlo simulation which is computationally intensive in low false-alarm regimes.

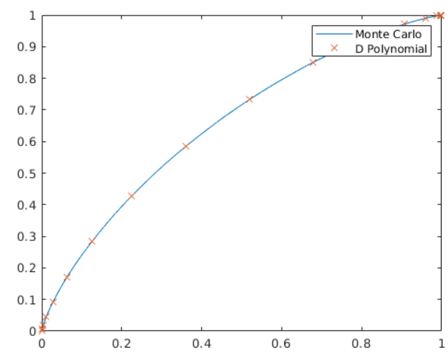

Fig. 3. P_f vs P_d computed using exact expressions for $F_{\lambda_1}(x)$ calculated using equation (4) and a Monte Carlo simulation with 10^6 trials for $M = 4$ and $N = 10^5$ with -15 dB per channel SNR

VI. Conclusion

This paper demonstrates new methods for tractable numerical computation of the distribution of the largest eigenvalue of a non-central complex Wishart matrix. Motivation for this problem comes from detection of a rank-one signal in Gaussian noise in multi-channel passive radar, where the largest eigenvalue of the Gram matrix constructed from data has this distribution and is the GLRT. The methods presented here extend previous state of the art in the non-central distribution corresponding to the signal present hypothesis. This enables numerical computation of the distribution with the large number of samples encountered in multi-channel sensing problems, allowing for exact calculation of ROC curves, which could previously only be calculated using Monte Carlo methods.

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