A Geometric View of Multistatic Radar Detection

Stephen D. Howard[1], Songsri Sirianunpiboon[1] and Douglas Cochran[2]
[1]Defence Science and Technology Group, PO Box 1500, Edinburgh 5111, Australia
[2]School of Mathematical and Statistical Sciences, Arizona State University, Tempe AZ 85287-5706 USA

***Abstract*—Coherent detection for multistatic radar is generally described in ways that obscure its underlying geometrical structure. This paper presents a view of multistatic radar in which the radar transmission essentially labels position-velocity space with subspaces. The detector for a target with a particular position and velocity is thus a matched subspace detector applied to the received signals. Within this framework, the ambiguity function takes a relatively simple form involving relative path losses. Furthermore, it is shown to be bounded by a function which purely geometric, in that it depends only on the waveforms and the geometry of the multistatic radar.**

***Index Terms*—Multistatic radar, MIMO radar, Multistatic radar detection, Multistatic ambiguity function, Geometric methods**

I. Introduction

Multistatic radar, or as it is often called these days, widely distributed MIMO radar, has received a great deal of attention in recent years [1], [2], [3], [4], [5], [6]. Due to the permutations of transmitter-target-receiver paths along with the use of multiple waveforms, multistatic radar is apt to be complicated. The way in which this problem has been formulated in the past has often obscured its underlying geometrical structure. This paper aims to improve the situation by developing an inherently geometric formulation of detection for multistatic radars. The geometric formulation is based on the view that the multistatic radar transmission labels each point in position-velocity space, or phase space, with a unique subspace. This is the subspace spanned by the particular set of delay and Doppler shifted transmit waveforms illuminating any possible target with that position and velocity. In this view the optimal detector for a single target in Gaussian white noise, is a subspace detector applied to the appropriately shifted received signals.

The ambiguity function associated with a waveform, or more correctly with the one dimensional subspace spanned by the waveform, has a long history in monostatic radar. In that context the ambigutiy function fulfills three roles. It is the point spread function for the minimum norm regularized radar image. It is the Kullback-Leibler divergence [7] for the estimation of range and Doppler of a single target in Gaussian white noise [8]. Finally, the ambiguity function can be interpreted as a measure of coupling of the optimal single target detector (for Gaussian white noise), for a target at a specific range and Doppler to an interfering target at another range and Doppler. It is this last interpretation which is important for detection.

In multistatic radar, there is no longer a single function which fulfills these three roles. This has led to a number of distinct, and often incompatable definitions of multistatic and MIMO ambiguity functions in the literature [4], [9], [10], [6]. In this paper, which deals with detection, the function which fulfills the third role is of direct interest. This is simpler in form than the position-velocity estimation ambiguity derived in [10], [6], which are based on [8]. Here a relatively simple geometric form of the detection ambiguity is derived. This result depends on the relative path losses of the transmit waveforms. The necessary inclusion of relative path losses in the detection ambiguity function is an additional complication when one wants to understand the performance of a particular set of waveforms, or transmit-receive geometry. An important result of the methods developed in this paper is that the ambiguity function can be bounded by a function which is purely geometric, in that it depends only on the waveforms and the geometry of the multistatic radar and not on the path losses. This is referred to as the *geometric ambiguity* and directly relates to the distance between two particular subspace.

II. Multistatic Radar Model

Consider a multistatic radar with N transmitters and M receivers. Nothing special is assumed about the configuration. Consider for the time being just the transmitters. The N transmitters each emit a different signal $s_j(t)$. The transmitted waveforms are taken to be elements of the Hilbert space $L^2(\mathbb{R})$, which has inner product

$$\langle s_1, s_2\rangle = \int_{\mathbb{R}} \overline{s_1(t)} s_2(t)\, dt.$$

Phase space Ω consists of all position velocity pairs $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{v})$, where $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^3$. The operators $D(\boldsymbol{p})$ for $\boldsymbol{p} \in \Omega$ are defined, for any $s \in L^2(\mathbb{R})$, by

$$D(\boldsymbol{p})s(t) = e^{i(\boldsymbol{v}\cdot\boldsymbol{x})(\omega/2c^2)} s(t - \|\boldsymbol{x}\|/c) e^{-i(\boldsymbol{v}\cdot\hat{\boldsymbol{x}})\omega t/c}.$$

In the context of radar, the constant c in this definition denotes the speed of light. The adjoint operator $D(\boldsymbol{p})^\dagger$, which corresponds physically to time reversal, is given by

$$D(\boldsymbol{p})^\dagger s(t) = e^{i(\boldsymbol{v}\cdot\boldsymbol{x})(\omega/2c^2)} s(t + \|\boldsymbol{x}\|/c) e^{i(\boldsymbol{v}\cdot\hat{\boldsymbol{x}})\omega t/c}.$$

It is obvious that $||D(\boldsymbol{p})s|| = ||s||$, for any $s \in L^2(\mathbb{R})$, so $D(\boldsymbol{p})$ is unitary and $D(\boldsymbol{p})^\dagger = D(\boldsymbol{p})^{-1}$.

The positions and velocities of transmitters and receivers in a multistatic scenario can all be specified by points in Ω. Take the transmitters to be at $\{\boldsymbol{t}_1, \cdots, \boldsymbol{t}_N\}$, and the receivers at $\{\boldsymbol{r}_1, \cdots, \boldsymbol{r}_M\}$.

Now consider the single-target detection problem; i.e., testing the hypothesis $H_1(\boldsymbol{p})$ that there is a target at the point

$\boldsymbol{p} \in \Omega$ against the hypothesis H_0 that the received signal is just receiver noise. The noise-free signal at receiver i under $H_1(\boldsymbol{p})$ is $D(\boldsymbol{r}_i - \boldsymbol{p})$ is

$$w_i = A\rho_{\boldsymbol{p},i} \sum_{j=1}^{N} \eta_{\boldsymbol{p},ij}\kappa_{\boldsymbol{p},j} D(\boldsymbol{p} - \boldsymbol{t}_j) s_j \tag{1}$$

where the meaning of the terms is a follows. In propagating from the transmitters to the point $\boldsymbol{p}$, each unit energy signal s_j, is transmitted with a certain actual energy which is then modified by antenna gains and path loss. An important aspect in detection and ambiguity analysis are the relative gains, which are the quantities $\kappa_{\boldsymbol{p},j}$. These satisfy

$$\sum_{j=1}^{N} |\kappa_{\boldsymbol{p},j}|^2 = 1.$$

Similarly, the $\rho_{\boldsymbol{p},i}$ represent the relative losses for propagation between $\boldsymbol{p}$ and receiver i. They also satisfy

$$\sum_{j=1}^{M} |\rho_{\boldsymbol{p},j}|^2 = 1.$$

The $\eta_{\boldsymbol{p},ij}$ are the scattering coefficients (cross section) of the target, and contain no path loss components. The prior probability distribution for the $\eta_{\boldsymbol{p},ij}$ is taken as Gaussian with $\mathsf{E}\{\eta_{\boldsymbol{p},ij}\}$ and $\mathsf{E}\{\eta_{\boldsymbol{p},ij}\overline{\eta_{\boldsymbol{p},i'j'}}\} = \beta_{\boldsymbol{p}}^2 \delta_{ii'}\delta_{jj'}$, where $\beta_{\boldsymbol{p}}^2$ may or may not be known. This prior is referred to as a statistically isotropic target model. The rest of the complex coefficients, which are the same for each transmit/receive pair, are collected in A. The signal received at receiver i is

$$x_i = D(\boldsymbol{r}_i - \boldsymbol{p}) w_i + n_i$$

where the n_i are zero mean, independent Gaussian white noise processes with

$$\mathsf{E}\{n_i(t)\overline{n_{i'}(t')}\} = \sigma^2 \delta_{ii'} \delta(t - t').$$

III. Geometric View of Multistatic Radar

In a nutshell, the geometric view of multistatic radar is that the multistatic transmission associates a unique subspace of $L^2(\mathbb{R})$ with each point $\boldsymbol{p} \in \Omega$. That is, transmission labels phase space with subspaces. A detector for a target at $\boldsymbol{p}$ is then designed by matching to the subspace associated with the point $\boldsymbol{p}$. In this and the next section, the details of this view are elucidated.

The set of normalized waveforms arriving at the point $\boldsymbol{p} \in \Omega$ is

$$S_{\boldsymbol{p}} = \{D(\boldsymbol{p} - \boldsymbol{t}_j)s_j \,|\, j = 1, \cdots, N\}$$

Note that we will often use the short hand $s_{\boldsymbol{p},j} = D(\boldsymbol{p} - \boldsymbol{t}_j)s_j$. The linear span

$$\mathcal{V}_{\boldsymbol{p}} = \operatorname{span} S_{\boldsymbol{p}}$$

is the set of all complex linear combinations of these waveforms and is a subspace of $L^2(\mathbb{R})$. The subspaces $\mathcal{V}_{\boldsymbol{p}}$ for $\boldsymbol{p} \in \Omega$ are fundamental in the multistatic radar detection problem. If the set of waveforms $S_{\boldsymbol{p}}$ is linearly independent, which it will be in general, then $\mathcal{V}_{\boldsymbol{p}}$ has dimension N. Associated uniquely with $\mathcal{V}_{\boldsymbol{p}}$ is the orthogonal projection onto $\mathcal{V}_{\boldsymbol{p}}$, which will be denoted by $\Pi_{\boldsymbol{p}}$. The operator $\Pi_{\boldsymbol{p}}$ is hermitian (self adjoint) and satisfies

$$\Pi_{\boldsymbol{p}}^2 = \Pi_{\boldsymbol{p}}, \quad \operatorname{Tr}(\Pi_{\boldsymbol{p}}) = N.$$

The projection $\Pi_{\boldsymbol{p}}$ can be constructed from $S_{\boldsymbol{p}}$, which forms a basis of unit vectors for $\mathcal{V}_{\boldsymbol{p}}$, in a standard way. In order to do this, it is useful to introduce the concept of a Gram matrix [11]. Given a set of L waveforms $W = \{w_j \in L^2(\mathbb{R}), j = 1, \cdots, L\}$, their $L \times L$ Gram matrix has elements

$$G(W)_{ij} = \langle w_i, w_j \rangle.$$

$G(W)$ is a positive semi-definite hermitian matrix for any set W, and it is positive definite if and only if W is a linearly independent set. This idea can be extended to two sets of waveforms, $W_1 = \{w_j^{(1)} | j = 1, \cdots, L_1\}$ and $W_2 = \{w_j^{(2)} | j = 1, \cdots, L_2\}$, to define the $L_1 \times L_2$ cross-Gram matrix with elements

$$G(W_1, W_2)_{ij} = \langle w_i^{(1)}, w_j^{(2)} \rangle.$$

Returning now to the construction of $\Pi_{\boldsymbol{p}}$ from the basis $S_{\boldsymbol{p}}$, the Gram matrix for $S_{\boldsymbol{p}}$ is

$$G_{\boldsymbol{p}} = G(S_{\boldsymbol{p}})$$

Corresponding to the basis $S_{\boldsymbol{p}}$, construct the dual basis [12]

$$\tilde{s}_{\boldsymbol{p},j} = \sum_{k=1}^{N} [G_{\boldsymbol{p}}^{-1}]_{kj}\, s_{\boldsymbol{p},k}, \quad j = 1, \cdots, N$$

which satisfies $\langle s_{\boldsymbol{p},i}, \tilde{s}_{\boldsymbol{p},j} \rangle = \delta_{ij}$ Then $\Pi_{\boldsymbol{p}}$ is defined by its action on any $w \in L^2(\mathbb{R})$,

$$\begin{aligned} \Pi_{\boldsymbol{p}} w &= \sum_{j=1}^{N} \langle \tilde{s}_{\boldsymbol{p},j}, w \rangle s_{\boldsymbol{p},j} \\ &= \sum_{j=1}^{N} \sum_{\ell=1}^{N} \left[G_{\boldsymbol{p}}^{-1}\right]_{j\ell} \left\langle D(\boldsymbol{p} - \boldsymbol{t}_j)s_\ell, w \right\rangle D(\boldsymbol{p} - \boldsymbol{t}_i)s_j. \end{aligned}$$

To summarise, the multistatic transmission allows each possible target position and velocity (point in $\boldsymbol{p} \in \Omega$) to be associated with a subspace $\mathcal{V}_{\boldsymbol{p}}$, defined by the linear span of the set $S_{\boldsymbol{p}}$. In the next section, testing the hypothesis that there is a target at $\boldsymbol{p}$, becomes one of testing for the presence of waveforms from $\mathcal{V}_{\boldsymbol{p}}$ in the appropriately transformed receive channels. The matched subspace detector is defined in terms of $\Pi_{\boldsymbol{p}}$.

Another quantity which is important in understanding detection is the following operator attached to the point $\boldsymbol{p}$, which is defined by its action on any $w \in L^2(\mathbb{R})$:

$$F_{\boldsymbol{p}} w = \sum_{j=1}^{N} |\kappa_{\boldsymbol{p},j}|^2 \langle s_{\boldsymbol{p},j}, w \rangle s_{\boldsymbol{p},j}. \tag{2}$$

Note that, for all $w \in L^2(\mathbb{R})$, $F_{\boldsymbol{p}} w \in \mathcal{V}_{\boldsymbol{p}}$. In terms of the one-dimensional projections $P_{\boldsymbol{p},j}$ on the $s_{\boldsymbol{p},j}$, defined by

$$P_{\boldsymbol{p},j} w = \langle s_{\boldsymbol{p},j}, w \rangle\, s_{\boldsymbol{p},j}$$

for $w \in L^2(\mathbb{R})$, (2) can be written as

$$F_{\boldsymbol{p}} = \sum_{j=1}^{N} |\kappa_{\boldsymbol{p},j}|^2 P_{\boldsymbol{p},j}. \tag{3}$$

The operator $F_{\boldsymbol{p}}$ is, in fact, the frame operator [12] of the set of waveforms $\{\kappa_{\boldsymbol{p},j} s_{\boldsymbol{p},j} \mid j = 1, \cdots, N\}$.

With the notation

$$\Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') = D(\boldsymbol{p} - \boldsymbol{r})^{\dagger} D(\boldsymbol{r} - \boldsymbol{p}'),$$

define the set of unit-energy waveforms

$$S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} = \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') S_{\boldsymbol{p}'} \equiv \{\Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') w \mid w \in S_{\boldsymbol{p}'}\}.$$

Note that if $\boldsymbol{p}' = \boldsymbol{p}$, then $\Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}) = I$. The set of waveforms $S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'}$ can be interpreted as follows. When the waveforms in $S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'}$ are propagated from $\boldsymbol{p}$ to $\boldsymbol{r}$, they produce the same set as when the waveforms in $S_{\boldsymbol{p}'}$ are propagated from $\boldsymbol{p}'$ to $\boldsymbol{r}$; i.e.,

$$D(\boldsymbol{r} - \boldsymbol{p}) S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} = D(\boldsymbol{r} - \boldsymbol{p}') S_{\boldsymbol{p}'}.$$

This set of waveforms generates the subspace

$$\mathcal{V}^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} = \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') \mathcal{V}_{\boldsymbol{p}'}$$

which has orthogonal projection

$$\Pi^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} = \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') \Pi_{\boldsymbol{p}'} \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}')^{\dagger}.$$

The operator $F_{\boldsymbol{p}'}$ can also be translated to $\boldsymbol{p}$ through $\boldsymbol{r}$ as

$$F^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} = \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') F_{\boldsymbol{p}'} \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}')^{\dagger}. \tag{4}$$

For set of points $R = \{\boldsymbol{r}_1, \cdots, \boldsymbol{r}_M\} \subset \Omega$, denote the union of the sets $S^{\boldsymbol{r}_i}_{\boldsymbol{p},\boldsymbol{p}'}$, $i = 1, \cdots, M$ by

$$S^{R}_{\boldsymbol{p},\boldsymbol{p}'} = \cup_{\boldsymbol{r} \in R} S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'}.$$

The subspace spanned by the waveforms in $S^{R}_{\boldsymbol{p},\boldsymbol{p}'}$ is the sum

$$\mathcal{V}^{R}_{\boldsymbol{p},\boldsymbol{p}'} = \sum_{\boldsymbol{r} \in R} \mathcal{V}^{r}_{\boldsymbol{p},\boldsymbol{p}'} = \sum_{\boldsymbol{r} \in R} \Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}') \mathcal{V}_{\boldsymbol{p}'}.$$

Recall that the sum of two subspaces is the linear span of the union of the two the subspaces. Denote the orthogonal projection onto $V^{R}_{\boldsymbol{p},\boldsymbol{p}'}$ by $\Pi^{R}_{\boldsymbol{p},\boldsymbol{p}'}$. In general, there is no simple relationship between $\Pi^{R}_{\boldsymbol{p},\boldsymbol{p}'}$ and the projections $\Pi^{\boldsymbol{r}_i}_{\boldsymbol{p},\boldsymbol{p}'}$. The frame operator $F^{R}_{\boldsymbol{p},\boldsymbol{p}'}$ is, however, defined as

$$\begin{aligned} F^{R}_{\boldsymbol{p},\boldsymbol{p}'} &= \sum_{i=1}^{M} |\rho_{\boldsymbol{p}',i}|^2 F^{\boldsymbol{r}_i}_{\boldsymbol{p},\boldsymbol{p}'} \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} |\rho_{\boldsymbol{p}',i}|^2 |\kappa_{\boldsymbol{p}',j}|^2 \Delta^{\boldsymbol{r}_i}(\boldsymbol{p}, \boldsymbol{p}') P_{\boldsymbol{p}',j} \Delta^{\boldsymbol{r}_i}(\boldsymbol{p}, \boldsymbol{p}')^{\dagger}. \end{aligned}$$

Note that $F^{R}_{\boldsymbol{p},\boldsymbol{p}'}$ is the frame operator of the set of waveforms

$$\{\rho_{\boldsymbol{p}',i} \kappa_{\boldsymbol{p}',j} \Delta^{\boldsymbol{r}_i}(\boldsymbol{p}, \boldsymbol{p}') s_{\boldsymbol{p}',j} | i = 1, \cdots, M,\ j = 1, \cdots, N\}.$$

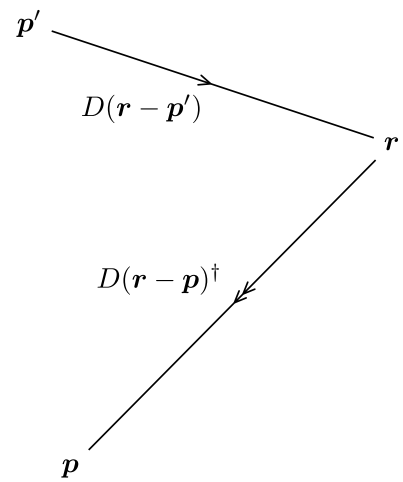

Fig. 1. Diagram representing the action of $\Delta^{\boldsymbol{r}}(\boldsymbol{p}, \boldsymbol{p}')$

Finally, some notation for the various Gram and cross-Gram matrices corresponding to the above sets is introduced:

$$\begin{aligned} G^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} &= G(S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'}) \\ G^{R}_{\boldsymbol{p},\boldsymbol{p}'} &= G(S^{R}_{\boldsymbol{p},\boldsymbol{p}'}) \\ \mathcal{X}^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'} &= G(S_{\boldsymbol{p}}, S^{\boldsymbol{r}}_{\boldsymbol{p},\boldsymbol{p}'}) \\ \mathcal{X}^{R}_{\boldsymbol{p},\boldsymbol{p}'} &= G(S_{\boldsymbol{p}}, S^{R}_{\boldsymbol{p},\boldsymbol{p}'}). \end{aligned}$$

IV. The Geometry of Detection

With the framework introduced in the preceding sections, testing the hypothesis that there is a target at $\boldsymbol{p}$ becomes one of testing for the presence of waveforms from $\mathcal{V}_{\boldsymbol{p}}$ in the appropriately transformed receive channels. To understand how this comes about, hypothesise that there is a single target at $\boldsymbol{p}$ and consider the signal at the i^{th} receiver, which is at $\boldsymbol{r}_i \in \Omega$. The signal at the receiver is

$$x_i = D(\boldsymbol{r}_i - \boldsymbol{p}) w_i + n_i$$

where $w_i \in \mathcal{V}_{\boldsymbol{p}}$ is the scattered waveform in the direction of receiver i, multiplied by any path loss in propagating to the receiver, and n_i is a Gaussian white noise (GWN) process. This can be rewritten as

$$D(\boldsymbol{r}_i - \boldsymbol{p})^{\dagger} x_i = w_i + n'_i$$

where n'_i is still a GWN process because $D(\boldsymbol{r}_i - \boldsymbol{p})$ is unitary. Now apply $\Pi_{\boldsymbol{p}}$, the orthogonal projector onto $\mathcal{V}_{\boldsymbol{p}}$, to this equation:

$$\Pi_{\boldsymbol{p}} D(\boldsymbol{r}_i - \boldsymbol{p})^{\dagger} x_i = w_i + \tilde{n}_i$$

where $\tilde{n}_i$ is now a GWN vector in the N-dimensional subspace $\mathcal{V}_{\boldsymbol{p}}$. Assuming nothing about the scattering process that produces the w_i, the optimal unitarily invariant detector is the energy detector with detection statistic

$$\begin{aligned}\gamma_{\boldsymbol{p}} &= \frac{1}{\sigma^2}\sum_{i=1}^{M}\langle x_i, D(\boldsymbol{r}_i-\boldsymbol{p})\Pi_{\boldsymbol{p}} D(\boldsymbol{r}_i-\boldsymbol{p})^\dagger x_i\rangle \\ &= \frac{1}{\sigma^2}\sum_{i=1}^{M}\|\Pi_{\boldsymbol{p}} D(\boldsymbol{r}_i-\boldsymbol{p})^\dagger x_i\|^2.\end{aligned}$$

In the Appendix, it is shown that the Bayesian log-likehood ratio with a non-informative prior on the scattering coefficients of the target is $\gamma_{\boldsymbol{p}}$.

V. Ambiguity Functions

As discussed in the introduction, in multistatic radar there is no longer a single function which fulfills all three roles that the monostatic ambiguity function does for monostatic radar. The ambiguity functions role in understanding detection is of interest here. That is, as a measure of coupling of the optimal single target detector (for Gaussian white noise), for a target at a specific position and velocity $\boldsymbol{p}$, to an interfering target at another position and velocity $\boldsymbol{p}'$.

The ambiguity function for the multistatic detection problem is

$$\begin{aligned}&\mathsf{E}_{H_1(\boldsymbol{p}')}\left\{\log\frac{p(Y|H_1(\boldsymbol{p}))}{p(Y|H_0)}\right\} - \mathsf{E}_{H_0}\left\{\log\frac{p(Y|H_1(\boldsymbol{p}))}{p(Y|H_0)}\right\} \\ &= \frac{1}{\sigma^2}\sum_{i=1}^{M}\mathsf{E}_{H_1(\boldsymbol{p}')}\left\{\langle D(\boldsymbol{r}_i-\boldsymbol{p})\Pi_{\boldsymbol{p}} D(\boldsymbol{r}_i-\boldsymbol{p})^\dagger x_i, x_i\rangle\right\} \\ &= |A|^2\frac{\beta_{\boldsymbol{p}'}^2}{\sigma^2}\operatorname{Tr}\left(\Pi_{\boldsymbol{p}} F_{\boldsymbol{p},\boldsymbol{p}'}^R\right).\end{aligned}$$

This implies that the multistatic ambiguity function can be defined by

$$\chi(\boldsymbol{p},\boldsymbol{p}') = \operatorname{Tr}\left(\Pi_{\boldsymbol{p}} F_{\boldsymbol{p},\boldsymbol{p}'}^R(\rho_{\boldsymbol{p}'},\kappa_{\boldsymbol{p}'})\right)$$

where the dependence of $F_{\boldsymbol{p},\boldsymbol{p}'}^R$ on $\rho_{\boldsymbol{p}'}$ and $\kappa_{\boldsymbol{p}'}$ has been noted explicitly. In terms of complexity this should be compared with the position-velocity estimation ambiguity derived in [6, Eq. (30)].

An important bound on the multistatic ambiguity can be proved using the theory of convex combinations of orthogonal projections [13]. The operator $F_{\boldsymbol{p},\boldsymbol{p}'}^R$ can be considered to be an operator on $\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R$, as ${\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^{\mathcal{R}}}^\perp$, the orthogonal complement of $\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R$, is in the null space of $F_{\boldsymbol{p},\boldsymbol{p}'}^R$ while the range of $F_{\boldsymbol{p},\boldsymbol{p}'}^R$ is contained in $\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R$. The results in [13] imply that, restricted to $\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R$,

$$F_{\boldsymbol{p},\boldsymbol{p}'}^R\big|_{\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R} \le I_{\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R}.$$

Thus,

$$F_{\boldsymbol{p},\boldsymbol{p}'}^R \le \Pi_{\boldsymbol{p},\boldsymbol{p}'}^R$$

and so the multistatic ambiguity function satisfies the bounding inequality

$$\chi(\boldsymbol{p},\boldsymbol{p}') = \operatorname{Tr}\left(\Pi_{\boldsymbol{p}} F_{\boldsymbol{p},\boldsymbol{p}'}^R(\rho_{\boldsymbol{p}'},\kappa_{\boldsymbol{p}'})\right) \le \operatorname{Tr}\left(\Pi_{\boldsymbol{p}}\Pi_{\boldsymbol{p},\boldsymbol{p}'}^R\right).$$

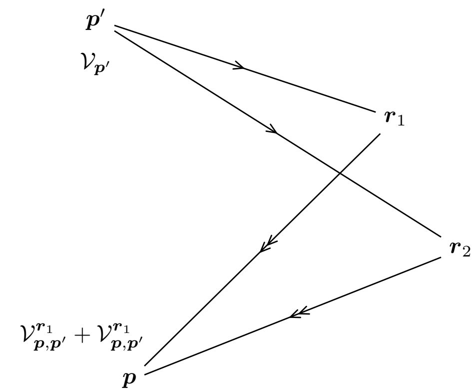

Fig. 2. The subspace at $\boldsymbol{p}$ corresponding to $\mathcal{V}_{\boldsymbol{p}'}$ at $\boldsymbol{p}'$ propagated through a pair of receivers

The quantity $\operatorname{Tr}\left(\Pi_{\boldsymbol{p}}\Pi_{\boldsymbol{p},\boldsymbol{p}'}^R\right)$ does not depend on $\rho_{\boldsymbol{p}'}$ or $\kappa_{\boldsymbol{p}'}$; rather it depends only on the unit transmit waveforms and the phase space geometry of the multistatic radar. We refer to this quantity as the *geometric ambiguity function*

$$\chi_G(\boldsymbol{p},\boldsymbol{p}') \equiv \operatorname{Tr}\left(\Pi_{\boldsymbol{p}}\Pi_{\boldsymbol{p},\boldsymbol{p}'}^R\right).$$

Choosing waveforms that make $\chi_G(\boldsymbol{p},\boldsymbol{p}')$ small in particular situations ensures that $\chi(\boldsymbol{p},\boldsymbol{p}')$ is also small. The geometric ambiguity is related to the distance between subspaces. A distance function between subspaces can the defined in terms of the Hilbert-Schmidt norm [14] of the difference of their corresponding orthogonal projectors as

$$\begin{aligned}d(\mathcal{V}_1,\mathcal{V}_2)^2 &= \|\Pi_{\mathcal{V}_1}-\Pi_{\mathcal{V}_2}\|_{\text{HS}}^2 \\ &= \dim(\mathcal{V}_1)+\dim(\mathcal{V}_2) - 2\operatorname{Tr}(\Pi_{\mathcal{V}_1}\Pi_{\mathcal{V}_2}).\end{aligned}$$

When the waveform set is linearly independent, $\dim(\mathcal{V}_{\boldsymbol{p}}) = N$, $\dim(\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R) = NM$, and it can be seen that the $\chi_G(\boldsymbol{p},\boldsymbol{p}')$ is directly related to the distance between $\mathcal{V}_{\boldsymbol{p}}$ and $\mathcal{V}_{\boldsymbol{p},\boldsymbol{p}'}^R$. The further apart these two subspaces are, the smaller $\chi_G(\boldsymbol{p},\boldsymbol{p}')$ will be.

Using (4) and (3) and the Gram matrices defined at the end of Section III, it is seen that $\chi(\boldsymbol{p},\boldsymbol{p}')$ can be written as

$$\begin{aligned}\chi(\boldsymbol{p},\boldsymbol{p}') &= \sum_{i=1}^{M}\sum_{j=1}^{N}|\rho_{\boldsymbol{p}',i}|^2|\kappa_{\boldsymbol{p}',j}|^2\|\Pi_{\boldsymbol{p}}\Delta^{\boldsymbol{r}_i}(\boldsymbol{p},\boldsymbol{p}')s_{\boldsymbol{p}',j}\|^2 \\ &= \operatorname{Tr}\left(G_{\boldsymbol{p}}^{-1}\sum_{i=1}^{M}|\rho_{\boldsymbol{p}',i}|^2\mathcal{X}_{\boldsymbol{p},\boldsymbol{p}'}^{\boldsymbol{r}_i}K_{\boldsymbol{p}'}(\mathcal{X}_{\boldsymbol{p},\boldsymbol{p}'}^{\boldsymbol{r}_i})^\dagger\right)\end{aligned}$$

where $K_{\boldsymbol{p}'}$ is the diagonal matrix with diagonal entries $|\kappa_{\boldsymbol{p}',j}|^2$. The first form gives $\chi(\boldsymbol{p},\boldsymbol{p}')$ as the weighed sum of the energies of the elements of $S_{\boldsymbol{p},\boldsymbol{p}'}^R$, after projection onto $\mathcal{V}_{\boldsymbol{p}}$. The geometric ambiguity $\chi_G(\boldsymbol{p},\boldsymbol{p}')$ can be written in terms of Gram matrices as

$$\chi_G(\boldsymbol{p},\boldsymbol{p}') = \operatorname{Tr}\left(G_{\boldsymbol{p}}^{-1}\mathcal{X}_{\boldsymbol{p},\boldsymbol{p}'}^R(G_{\boldsymbol{p},\boldsymbol{p}'}^R)^{-1}(\mathcal{X}_{\boldsymbol{p},\boldsymbol{p}'}^R)^\dagger\right).$$

Appendix A
Bayesian Likelihood Ratio

As described in Section IV for a single target at $\boldsymbol{p} \in \Omega$ (hypothesis $H_1(\boldsymbol{p})$), the received signal at receiver i

$$x_i = D(\boldsymbol{r}_i - \boldsymbol{p}) w_i + n_i$$

where $w_i \in \mathcal{V}_{\boldsymbol{p}}$ is given by (1). If there is no target present (hypothesis H_0), then $x_i = n_i$.

In terms of the orthonormal basis

$$u_{\boldsymbol{p},\ell} = \sum_{j=1}^{N} [G_{\boldsymbol{p}}^{-1/2}]_{\ell,j}\, s_{\boldsymbol{p},j}, \quad \ell = 1, \cdots, N,$$

for $\mathcal{V}_{\boldsymbol{p}}$,

$$\begin{aligned} Y_{i\ell} &= \langle u_{\boldsymbol{p},\ell}, D(\boldsymbol{r}_i - \boldsymbol{p})^\dagger x_i \rangle \\ &= A\rho_{\boldsymbol{p},i} \sum_{j=1}^{N} \eta_{\boldsymbol{p},ij} \kappa_{\boldsymbol{p},j} [G_{\boldsymbol{p}}^{1/2}]_{\ell j} + \nu_{i\ell} \end{aligned}$$

where the $\nu_{i\ell}$ are zero-mean Gaussian distributed with $\mathsf{E}(\nu_{i\ell}\overline{\nu_{i'\ell'}}) = \sigma^2 \delta_{ii'} \delta_{\ell\ell'}$ Under H_1 the vectors Y_i are independently distributed with probability density

$$p(Y_i | H_1(\boldsymbol{p}), A, \boldsymbol{\eta}) = \frac{1}{(\pi\sigma^2)^{NM}} \exp -\frac{1}{\sigma^2} \| Y_i - A\rho_{\boldsymbol{p},i} \sum_{j=1}^{N} \eta_{\boldsymbol{p},ij} \kappa_{\boldsymbol{p},j} [G_{\boldsymbol{p}}^{1/2}]_{\ell j} \|_F^2 \tag{5}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

Our prior distribution for the scattering matrix $\eta_{\boldsymbol{p},ij}$, is that corresponding to a statistically isotropic target; i.e.,

$$p(\boldsymbol{\eta}_{\boldsymbol{p}} | \beta_{\boldsymbol{p}}^2) = \frac{1}{(\pi\beta_{\boldsymbol{p}}^2)^{NM}} \exp\left(-\frac{\|\boldsymbol{\eta}_{\boldsymbol{p}}\|^2}{\beta_{\boldsymbol{p}}^2} \right).$$

The prior becomes non-informative as $\beta_{\boldsymbol{p}} \to \infty$. Marginalizing (5) with respect to $\boldsymbol{\eta}$, the log-likelihood ratio for $H_1 : A > 0$ against $H_0 : A = 0$ is

$$\log \frac{p(Y | H_1(\boldsymbol{p}), A, \beta_{\boldsymbol{p}}^2)}{p(Y|H_0)} = -\frac{1}{\sigma^2} \sum_{i=1}^{M} Y_i \left(\left(\frac{|A|^2 \beta_{\boldsymbol{p}}^2}{\sigma^2} \rho_{\boldsymbol{p},i}^2 G_{\boldsymbol{p}}^{1/2} K G_{\boldsymbol{p}}^{1/2} + I \right)^{-1} - I \right) Y_i^\dagger.$$

In the limit $\beta_{\boldsymbol{p}} \to \infty$ this becomes

$$\begin{aligned} \log \frac{p(Y | H_1(\boldsymbol{p}))}{p(Y|H_0)} &= \frac{1}{\sigma^2} \sum_{i=1}^{M} Y_i Y_i^\dagger \\ &= \frac{1}{\sigma^2} \sum_{i=1}^{M} \langle x_i, D(\boldsymbol{r}_i - \boldsymbol{p}) \Pi_{\boldsymbol{p}} D(\boldsymbol{r}_i - \boldsymbol{p})^\dagger x_i \rangle \end{aligned}.$$

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