THE GEOMETRY OF COHERENCE AND ITS APPLICATION TO CYCLOSTATIONARY TIME SERIES

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ABSTRACT

The consequences of cyclostationary structure in a random process have traditionally been described in terms of the correlation or coherence of pairs of particular time and frequency shifted versions of the process. However, cyclostationarity, and more generally almost cyclostationarity, are manifest in the mutual coherence of subspaces spanned by sets of time and frequency shifted versions of the process. The generalized coherence framework allows any finite collection of pertinent samples of the cyclic autocorrelation function estimates formed from the measured signal data to be combined into a detection statistic. This paper develops the subspace coherence theory of almost cyclostationary processes as a guide to constructing such detectors in both the time and spectral domains.

***Index Terms*—** Cyclostationarity, Coherence, Multiple-channel detection

I. INTRODUCTION

A zero-mean random signal $x(t)$ is said to be second-order or wide-sense cyclostationary if its auto-correlation function, $C(t,t') = \mathsf{E}\{\overline{x(t)}x(t')\}$, is periodic with some period T in the sense that $C(t+T,t'+T) = C(t,t')$ for all t and t'. The concept of cyclostationarity models the subtle periodicity exhibited by many natural and artificial processes where the signal itself is not periodic due to random variation in components of the signal, but where the statistics of the signal do exhibit periodicities [1], [2], [3]. There has been renewed interest in cyclostationary detection in the past few years in connection with cognitive radio [4], [5], [6].

The problem of detecting a cyclostationary signal with a given cycle period embedded in wide-sense stationary (WSS) noise has a long history. Perhaps the most important property of cyclostationary signals is that correlation exists between the cyclostationary signal and an arbitrarily time-shifted copy of the signal modulated by the cycle frequency. Detectors originating from the work of Gardner [3], [7] use a single time lag, usually zero, and consider the above correlation across one or more spatial channels to detect cyclostationarity. The detectors of Dandawate and Giannakis [8], [9] use the estimated cyclic correlation function across multiple time lags. The joint asymptotic distribution of these correlations is used to construct a detection statistic.

In [10] the use of the magnitude-squared coherence (MSC) estimate was proposed as a measure of spectral correlation for cyclostationary detection. It was also suggested, without further development, that the generalized coherence (GC) statistic [11], [12] might be used for polycyclic signal detection.

An alternative approach to cyclosationary detection is based on generalized likelihood ratio test (GLRT) or locally most powerful invariant test (LMPIT) formulations [13], [14], [5]. These are derived in the Fourier domain, asymptotically for large signal length. These detectors are shown in [5] to have an advantage over previous detectors for MIMO signals in which there is very significant multipath mixing. It should be noted that statistics that have the structure of the generalized coherence statistics occur in [5]; in that work, however, coherence between data vectors is only taken over independent realizations.

In a recent paper [15] we developed a number of detectors for the presence of cyclosationarity against WSS noise. These are based on the generalized coherence framework which allows any finite collection of pertinent samples of the cyclic autocorrelation function estimates formed from the measured signal data to be combined into the detection statistic. These detectors where shown by simulation to have superior performance to established cyclostationarity detectors in a number of disparate scenarios. This paper has two goals. The first is to develop the subspace coherence theory of almost cyclostationary processes, as a guide to constructing such detectors in both the time and spectral domains. The second goal is to understand the detectors developed in [5] in this context.

The paper begins with a review of multiple coherence theory in a strongly geometric context. Section III develops the theory of multiple coherence as it relates to the structure of almost cyclostationary random processes. In Section IV the construction of detectors based on this theory is discussed, particularly in the spectral domain, which was not addressed in [15]. It is also shown that the insights gained, when applied to GLR/LMPIT detectors in [5], lead to modifications that despite violating the assumptions under which the detectors are derived, considerably improve their performance in a scenario in which they perform poorly in [15].

II. THE GEOMETRY OF COHERENCE

The coherence of sets of random variables is an inherently geometric concept. Consider the standard treatment of canonical correlations. Let $\mathcal{H}$ be the Hilbert space of complex-valued random variables on some probability space. The inner product between two random variables x and y in $\mathcal{H}$ is $\langle x,y\rangle = \mathsf{E}\{x\overline{y}\}$.

Consider two sets of K zero-mean random variables $C_X = \{x_1,\cdots,x_K\}$ and $C_Y = \{y_1,\cdots,y_K\}$. Associated with the sets C_X and C_Y are the subspaces V_X and V_Y of $\mathcal{H}$ consisting of the linear spans of the set of random variables. The canonical correlation coefficients [16], [17] between C_X and C_Y, denoted by ρ_j, $j=1,\cdots,K$, are the singular values of $R_{XX}^{-1/2}R_{XY}R_{YY}^{-1/2}$, where R_{XX} and R_{YY} are the covariance matrices of C_X and C_Y, respectively, and R_{XY} denotes their cross-covariance matrix. The canonical correlation coefficients are actually the cosines of the principal angles between the subspaces V_X and V_Y with respect to the above inner product. It has long been known that canonical correlations are the eigenvalues of the product $P_{V_X}P_{V_Y}$, where P_V denotes the orthogonal projection onto the subspace $V\subset\mathcal{H}$ [18],

[19]. The squared coherence between two subspaces V_X and V_Y of $\mathcal{H}$ is defined by

$$\gamma^2(V_X, V_Y) = 1 - \frac{\begin{vmatrix} R_{XX} & R_{XY} \\ R_{XY}^\dagger & R_{YY} \end{vmatrix}}{|R_{XX}||R_{YY}|}.$$

This does not depend on the respective bases of the subspaces.

Now consider the above from a purely geometric perspective. Take $\mathcal{H}$ to be a general separable Hilbert space. Geometrically the fundamental object in relation to coherence is the Gram matrix and, in particular, its determinant. Given a set of vectors $\{x_1, \cdots, x_K\}$ in $\mathcal{H}$, their Gram matrix G has entries $G_{ij} = \langle x_i, x_j \rangle$. The determinant $|G|$ is the square of the volume of the parallelotope formed by the vectors. In general the vectors can be associated with the bounded operator $X : \mathcal{H} \to \mathbb{C}^K$ which maps any $y \in \mathcal{H}$ to the vector $(\langle x_1, y\rangle, \cdots, \langle x_K, y\rangle)$. The Gram matrix is then $XX^\dagger$, where $\dagger$ denotes the Hilbert space adjoint.

To define coherence between two subspaces V_X and V_Y, first consider an orthonormal basis for V_Y. The associated bounded operator $U_Y : \mathcal{H} \to \mathbb{C}^{\dim V_Y}$ has the properties $G = U_Y U_Y^\dagger = I$ and $U_Y^\dagger U_Y = P_{V_Y}$. Since the basis is orthonormal the squared volume of the parallelotope formed by the basis vectors is 1. Now orthogonally project the orthonormal set of vectors onto $V_X^\perp$, the orthogonal complement of V_X. The squared volume of the parallelotope formed by the projected vectors is

$$|U_Y P_{V_X^\perp} U_Y^\dagger| = |I - U_Y P_{V_X} U_Y^\dagger| = |I - P_{V_X} P_{V_Y}|.$$

The squared coherence is defined as the discrepancy between the original volume and the volume after projection

$$\gamma^2(V_X, V_Y) = 1 - |I - P_{V_X} P_{V_Y}|. \tag{1}$$

The more the original rectangular parallelotope in V_Y collapses in volume when projected onto $V_X^\perp$ the greater the coherence of the two subspaces. In fact, if the projected volume is zero, then V_Y contains at least a one-dimensional subspace that is also contained in V_X. In statistical terms, this means that given the values of the random variables defining V_X, then for any random variable V_Y, its component along the common one-dimensional subspace can be predicted with probability 1.

Apart from its geometric simplicity, the form (1) has the advantage that it is directly applicable to infinite dimensional subspaces, such as those that occur when defining canonical correlations for random processes [20]. In the infinite dimensional case the determinant is defined according to [21].

In general the mutual coherence between a collection of M subspaces $\{V_j\} \equiv \{V_j | j = 1, \cdots, M\}$ is given by

$$\gamma^2(\{V_j\}) = 1 - \prod_{j=1}^{N} |I - P_{\sum_{\ell=1}^{j} V_\ell} P_{V_{j+1}}| \tag{2}$$

where on the right hand side the subspaces can be indexed in any order. For three subspaces this takes the form

$$\gamma^2(V_X, V_Y, V_Z) = 1 - |I - P_{V_X} P_{V_Y}||I - P_{V_X + V_Y} P_{V_Z}|$$

so that computing the coherence decomposes into computing the coherence between $V_X + V_Y$ and V_Z and then the coherence between V_X and V_Y.

In terms of a choice of orthonormal bases U_j for the subspaces V_j a Gram matrix can be associated with the set of subspaces

$$\widetilde{G}(\{V_j\}) = \begin{pmatrix} I & U_1 U_2^\dagger & \dots & U_1 U_N^\dagger \\ U_2 U_1^\dagger & I & \dots & U_2 U_N^\dagger \\ \vdots & \vdots & & \vdots \\ U_N U_1^\dagger & U_N U_2^\dagger & \dots & I \end{pmatrix}.$$

The mutual coherence can then be written as

$$\gamma^2(\{V_j\}) = 1 - |\widetilde{G}(\{V_j\})|.$$

Another measure of coherence which has arisen in the recent work of Ramírez [22], [5] on locally most powerful invariant tests (LMPIT) for correlation. These are derived in the weak coherence limit and take the form of the Frobenius, or Hilbert-Schmidt, norm $\|\widetilde{G} - I\|_F$. The relationship of this to the above definition can be deduced using the operator expansion

$$|I - \mu A| = \sum_{n=0}^{\infty} \mu^n \mathrm{Tr}(\wedge^n A)$$

where A for a matrix or suitably well behaved operator on a Hilbert space [21] and $\wedge^n A$ denotes the n^{th} exterior power of A. Applying this expansion to the normalized Gram matrix form of the mutual coherence of a collection of subspaces gives

$$\gamma^2(\{V_j\}) = 1 - |\widetilde{G}| = -\mathrm{Tr}(\wedge^2(\widetilde{G} - I)) + \text{higher order terms.}$$

Noting that $\mathrm{Tr}(\wedge^2 A) = \sum_{i<j} (A_{ii}A_{jj} - A_{ji}A_{ij})$, the weak coherence limit of the mutual coherence becomes

$$\gamma^2(\{V_j\}) \sim \frac{1}{2}\|\widetilde{G} - I\|_F^2 = \sum_{i<j} \mathrm{Tr}\left(P_{V_i} P_{V_j}\right)$$

This can be compared with (2). The limiting form of the squared coherence will be denoted by $\gamma_F^2(\{V_j\})$ below.

III. COHERENCE AND ALMOST CYCLOSTATIONARITY

The goal for this section is first to set up an appropriate Hilbert space of random process in which to properly formulate coherence in the context of almost cyclostationary processes. These are processes for which the autocorrelation $C(t+\tau, t)$ is an almost periodic function of t for every value of τ [1]. Most of what is needed for this task can be found in [23] and the references therein.

Take a random complex process $x(t)$ on $\mathbb{R}$ with autocorrelation $C(t, t')$ and write $B(t, \tau) = C(t + \tau, t)$. Assume that $B(., \tau) \in L^1_{\text{loc}}(\mathbb{R})$, for every $\tau \in \mathbb{R}$, and that

$$\lim_{Z\to\infty} \frac{1}{2Z} \int_{-Z+c}^{Z+c} B(t, \tau) e^{i\omega t} dt \tag{3}$$

exists independently of c, for every τ and $\omega \in \mathbb{R}$ (See [23]). This condition is satisfied for all almost cyclostationary processes.

Now for a fixed value of $Z \in \mathbb{R}$, consider the Hilbert space H_x^Z generated by all the time and frequency translates $x_{\tau,\omega}(t) = x(t + \tau)e^{i\omega t}$, of x, restricted to the interval $t \in (-Z, Z)$, with inner product

$$\langle y_1, y_2 \rangle_Z = \frac{1}{2Z} \int_{-Z}^{Z} \mathsf{E}\left(y_1(t)\overline{y_2(t)}\right) dt.$$

The Hilbert space H_x^Z contains the short time Fourier transform (STFT)

$$\widetilde{X}_\alpha^W(t) = \frac{1}{\sqrt{W}} \int_{t-W/2}^{t+W/2} x(u) e^{-i\alpha u} du \tag{4}$$

for all $W \in (0, \infty)$.

If the process $x(t)$ is almost cyclostationary then

$$B(t, \tau) = \sum_{\omega\in\Omega} R_\omega(\tau) e^{i\omega t}$$

for some countable set of Fourier exponents $\omega \in \Omega$, where

$$R_\omega(\tau) = \lim_{Z\to\infty} \langle x_\tau,\ x_\omega \rangle_Z.$$

If $x(t)$ is cyclostationary with cyclic (angular) frequency α then the set $\Omega = \{n\alpha \,|\, n \in \mathbb{Z}\}$. Recall the corresponding cyclic spectral densities are defined as

$$S_\alpha(\omega) = \int_{-\infty}^{\infty} R_\alpha(\tau) e^{-i\omega\tau}\, d\tau. \tag{5}$$

For $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_K)$ consider the subspace $V_{\boldsymbol{\tau}}^Z$ of H_x^Z spanned by $x_{\tau_1}, \cdots, x_{\tau_K}$. Assume that the set $\boldsymbol{\tau}$ is chosen so that $V_{\boldsymbol{\tau}}^Z$ has dimension K, that is, $\{x_{\tau_1}, \cdots, x_{\tau_K}\}$ is a basis for its linear span. The Gram matrix $\mathbf{R}_{\boldsymbol{\tau}}^Z$, for the set $\{x_{\tau_1}, \cdots, x_{\tau_K}\}$ has components

$$[\mathbf{R}_{\boldsymbol{\tau}}^Z]_{ij} = \langle x_{\tau_i},\ x_{\tau_j}\rangle_Z.$$

Now consider the limit

$$\mathbf{R}_{\boldsymbol{\tau}} = \lim_{Z\to\infty} \mathbf{R}_{\boldsymbol{\tau}}^Z.$$

$\mathbf{R}_{\boldsymbol{\tau}}$ is non-negative definite as it is the limit of positive or non-negative definite matrices. Further, condition (3) implies that

$$[\mathbf{R}_{\boldsymbol{\tau}}]_{ij} = R_0(\tau_i - \tau_j).$$

Thus $R_0(\tau)$ is a non-negative definite function under condition (3). This was first shown in [23] and only requires that (3) be satisfied for $\omega = 0$ and all τ. Now consider the modulated subspace $V_{\boldsymbol{\tau},\omega}^Z$ spanned by $\{x_{\tau_1,\omega}, \cdots, x_{\tau_K,\omega}\}$. In a similar way to the above, for any value of ω, define the Gram matrix

$$[\mathbf{R}_{\boldsymbol{\tau},\omega}^Z]_{ij} = \langle x_{\tau_i},\ x_{\tau_j,\omega}\rangle_Z.$$

Condition (3) implies that

$$[\mathbf{R}_{\boldsymbol{\tau},\omega}]_{ij} = \lim_{Z\to\infty}[\mathbf{R}_{\boldsymbol{\tau},\omega}^Z]_{ij} = R_\omega(\tau_i - \tau_j).$$

Thus, the squared coherence between the subspaces $V_{\boldsymbol{\tau}}^Z$ and $V_{\boldsymbol{\tau},\omega}^Z$ is given by

$$\gamma^2(V_{\boldsymbol{\tau}}^Z, V_{\boldsymbol{\tau},\omega}^Z) = 1 - \left|I - P_{V_{\boldsymbol{\tau}}^Z} P_{V_{\boldsymbol{\tau},\omega}^Z}\right| = 1 - \frac{\begin{vmatrix}\mathbf{R}_{\boldsymbol{\tau}}^Z & \mathbf{R}_{\boldsymbol{\tau},\omega}^Z \\ \mathbf{R}_{\boldsymbol{\tau},\omega}^{Z\dagger} & \mathbf{R}_{\boldsymbol{\tau}}^Z\end{vmatrix}}{|\mathbf{R}_{\boldsymbol{\tau}}^Z|^2}$$

$$\xrightarrow[Z\to\infty]{} 1 - \frac{\begin{vmatrix}\mathbf{R}_{\boldsymbol{\tau}} & \mathbf{R}_{\boldsymbol{\tau},\omega} \\ \mathbf{R}_{\boldsymbol{\tau},\omega}^{\dagger} & \mathbf{R}_{\boldsymbol{\tau}}\end{vmatrix}}{|\mathbf{R}_{\boldsymbol{\tau}}|^2}$$

Thus the presence of a Fourier exponent of an almost cyclostationary process at α is manifest in the coherence of the subspaces V_{τ}^Z and $V_{\tau,\alpha}^Z$. The presence of cyclostationary, with cyclic frequency α, is manifest in multiple coherence within the set of subspaces $\{V_{\boldsymbol{\tau},n\alpha}^Z | n \in \mathbb{Z}\}$. For example, a test for cyclostationarity against WSS noise, using delays $\boldsymbol{\tau}$ and two harmonics of the cyclic frequency, can be based on the mutual coherence

$$\gamma^2(V_{\boldsymbol{\tau}}^Z, V_{\boldsymbol{\tau},\alpha}^Z, V_{\boldsymbol{\tau},2\alpha}^Z) \xrightarrow[Z\to\infty]{} 1 - \frac{\begin{vmatrix}\mathbf{R}_{\boldsymbol{\tau}} & \mathbf{R}_{\boldsymbol{\tau},\alpha} & \mathbf{R}_{\boldsymbol{\tau},2\alpha} \\ \mathbf{R}_{\boldsymbol{\tau},\alpha}^{\dagger} & \mathbf{R}_{\boldsymbol{\tau}} & \mathbf{R}_{\boldsymbol{\tau},\alpha} \\ \mathbf{R}_{\boldsymbol{\tau},2\alpha}^{\dagger} & \mathbf{R}_{\boldsymbol{\tau},\alpha}^{\dagger} & \mathbf{R}_{\boldsymbol{\tau}}\end{vmatrix}}{|\mathbf{R}_{\boldsymbol{\tau}}|^3}$$

One can then see the way in which coherence for modulated subspaces corresponding higher harmonics can be added.

The case of multichannel cyclostationarity can easily be incorporated into the above analysis by reinterpreting the subspace $V_{\boldsymbol{\tau}}$ to be the subspace generated by all of the channels and all of their individual times shifts by the elements of $\boldsymbol{\tau}$.

Now consider the short time Fourier transform (STFT) (4). It can be shown that

$$\begin{aligned}&\lim_{Z\to\infty} \langle \widetilde{X}_{\omega_1}^W, \widetilde{X}_{\omega_2}^W\rangle_Z \\ &= \frac{1}{W}\int_{-W/2}^{W/2}\int_{-W/2}^{W/2} R_{\omega_1-\omega_2}(\tau-\tau')e^{-i\omega_1(\tau-\tau')}d\tau d\tau'.\end{aligned} \tag{6}$$

If the process $x(t)$ is almost cyclostationary, then the right hand side of (6) is non-zero, if and only if $\omega_1 - \omega_2 \in \Omega$, the set of Fourier exponents. In the case of cyclostationarity with cyclic frequency α, the (complex) spectrogram components are correlated if and only if they are separated by an integer multiple of α. Note that this is true for all values of W, not just in the limit $W \to \infty$. If the limit $W \to \infty$ is taken, then, as shown in [10], comparison with (5) gives

$$\lim_{W,Z\to\infty} \langle \widetilde{X}_{\omega_1}^W, \widetilde{X}_{\omega_2}^W\rangle_Z = S_{\omega_1-\omega_2}(\omega_1).$$

Consider a cyclostationary process with cyclic frequency α. For any frequency $0 \le \omega < \alpha$, consider the collection of one dimensional subspaces $U_{\omega+n\alpha}^Z$, spanned by $\widetilde{X}_{\omega+n\alpha}$, for $n \in \mathbb{Z}$. If $\omega_1 \ne \omega_2$, then in the limit as $Z \to \infty$,

$$U_{\omega_1+n\alpha}^Z \perp U_{\omega_2+n'\alpha}^Z$$

for all $n, n' \in \mathbb{Z}$. Thus, cyclostationarity with cyclic frequency α, is indicated by mutual coherence within the set subspaces $A_{\omega,\alpha} = \{U_{\omega+n\alpha}^Z | n = 0, \cdots, K-1\}$, where K is any finite subset of $\mathbb{Z}$. In fact, the detector proposed in [10] exploits coherence between the subspaces U_ω^Z and $U_{\omega+\alpha}^Z$ for a single frequency ω. In addition, the cyclostationarity vs. WSS noise detectors in [5] can be interpreted as exploiting the coherence of larger subsets of $A_{\omega,\alpha}$, combined over several values of ω. The details of this will be discussed in the next section.

IV. CONSTRUCTION OF COHERENCE BASED DETECTORS OF CYCLOSTATIONARITY

Consider a multichannel data collection which consists of L channels of N data samples. The problem is to detect the presence of a cyclostationary signal, with cyclic frequency α, in one or more of the channels against WSS noise. Based on ideas presented in the previous section, we recently [15] proposed the following detector for this problem. The detector relies on choosing an appropriate set of time shifts $\boldsymbol{\tau}$ and number of harmonics of the cycle frequency under test. Denote by $\mathbf{X}_{\boldsymbol{\tau}}$, the matrix with rows consisting of all of the data channels and all of their individual time shifts by $\boldsymbol{\tau}$. Further, denote the modulated versions of $\mathbf{X}_{\boldsymbol{\tau}}$ by $\mathbf{X}_{\boldsymbol{\tau}}^{n_j\alpha}$, for integers $n_1, \cdots, n_H$. Write

$$\mathbf{Y} = \begin{pmatrix}\hat{\mathbf{X}}_{\boldsymbol{\tau}} \\ \vdots \\ \hat{\mathbf{X}}_{\boldsymbol{\tau}}^{n_H\alpha}\end{pmatrix}$$

where $\hat{}$ indicates that the rows of $\mathbf{X}_{\boldsymbol{\tau}}$ etc., have been orthonormalized. The multiharmonic coherence for the data subspaces $V_{\boldsymbol{\tau}}, \ldots, V_{\boldsymbol{\tau},n_H\alpha} \subset \mathbb{C}^N$ is given by

$$\gamma^2(V_{\boldsymbol{\tau}}, V_{\boldsymbol{\tau},n_1\alpha}, \ldots, V_{\boldsymbol{\tau},n_H\alpha}) = 1 - |\mathbf{Y}\mathbf{Y}^\dagger|.$$

This detector was shown in [15] to have superior performance to existing cyclostationarity detectors, including those proposed by Ramírez et al. [5], in a number of disparate examples. Here our interest is to develop spectral domain coherence detectors based on the analysis at the end of the previous section, and understand their relationship to the detectors proposed in [5] and [10].

Ramírez et al., derived GLR and LMPIT detectors for the above detection problem in a Gaussian setting. The detectors have the structure of coherence statistics. Their detectors are derived based on two requirements. The first involves strict assumptions on the sampling rate to ensure that there are an exact number of samples per cyclic period. The second assumption is that the signal consists of a number of equal size independent realizations, each an integer number of cyclic periods long. Each of these independent realizations are Fourier transformed. The data then takes the form like a discrete STFT in which individual "times" are guaranteed to be independent. The detector is then derived in

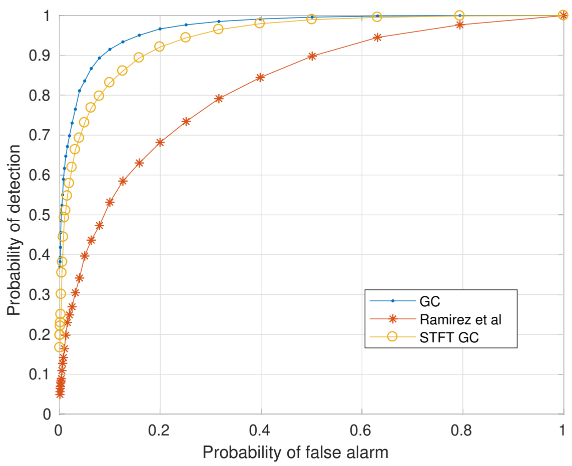

Fig. 1. ROC curves for cyclostationarity vs. WSS, $M = 4$, SNR = -16dB

the large window limit. Of course, a real signal does not arrive as a sequence of independent realizations, so that in practice the detector must be based on the actual STFT of the data, with non-overlapping windows.

Consider the relationship between this detector and the STFT coherence of the last section. For the moment consider a single channel of data from the above detection problem. Apply a STFT of window length W samples, to the data. Initially W is not assumed to have any relationship to cyclic frequency α, so the FFT cannot be used. Instead for a set of equi-spaced frequencies $0 \le \omega_j < \alpha$ and some set H of values n satisfying $n\alpha < 2\pi$ compute the STFT:

$$\widetilde{X}(\ell K, \omega_j + n\alpha) = e^{i(\omega_j+n\alpha)\ell K} \sum_{m=0}^{W-1} x_{\ell K+m} e^{-i(\omega_j+n\alpha)m}$$

where K is the undersampling factor and $\ell = 0, \cdots, L-1$ is the number of samples permitted by the data length. Consider the one dimensional subspaces $V_{\omega_j+n\alpha}$ of $\mathbb{C}^L$ each spanned by a single unit vector $\hat{X}_{\omega_j+n\alpha}$, the unit vector corresponding to the STFT channel at frequency $\omega_j + n\alpha$. In the presence of the cyclostationarity signal there should exist coherence between the subspaces $\{V_{\omega_j+n\alpha} | n \in H\}$, while there should exist no coherence between subspaces labeled by distinct values of ω_j under either hypothesis. For each ω_j construct the matrices

$$\mathbf{X}_{\omega_j} = \begin{pmatrix} \hat{X}_{\omega_j} \\ \hat{X}_{\omega_j+\alpha} \\ \hat{X}_{\omega_j+2\alpha} \\ \vdots \end{pmatrix}.$$

The mutual coherence of the subspaces $V_{\omega_j+n\alpha}$ for all ω_j and $n \in H$ is

$$\gamma^2(\{V_{\omega_j+n\alpha}\}) = 1 - \prod_j |\hat{\mathbf{X}}_{\omega_j} \hat{\mathbf{X}}^\dagger_{\omega_j}|.$$

Now the only change necessary if there are M channels of data is that each entry in matrix $\mathbf{X}_\omega$, for example, $\hat{X}_{\omega_j+\alpha}$ has M rows rather than 1 and the ˆ denotes that these rows have been orthonomalized.

Coming back to the conditions imposed in [5]. From the point of view of the coherence arguments presented here, the conditions just provide convenience of calculation. The constraints take the form that $\alpha = 2\pi/P$, for some integer P. The window length $W = N_s P$, for some integer N_S. This then implies that the FFT can be used to compute the STFT, the $\omega_j = 2\pi j/(N_s P)$, for $j = 0, \cdots, N_S - 1$ and $H = \{0, \cdots, P-1\}$. In general, one has the choice of directly computing the STFT in α/N_s spacing or resampling the data appropriately for each tested α.

Comparing the time-domain and STFT or spectral-domain (with non-overlapping windows) detectors described in this section, it can be seen that the time-domain detectors are based on a relatively small number of cross correlations over the entire length of the data N. On the other hand the spectral-domain detectors use a large number of cross correlations between spectral channels, but here the channels have a length of only N/W. Simulation results given in [15] in a number of scenarios suggest that the time-domain coherence based detectors have substantially better performance, when non-overlapping windows, or equivalently critical sampling is used in the STFT.

From the point of view of coherence detection critical sampling of the STFT seems an unnecessary limitation. One advantage of critical sampling, as pointed out in [10] for two spectral channels, is that when testing against white noise, the distribution of the coherence detector can computed under the null hypothesis. Obviously, over sampling the STFT in time, produces longer channel vectors. The question is what effect this has on the performance of the detector. To at least gain some indication of the effect of oversampling the STFT we performed the simulation described below (IV-A). The simulation tests for the presence of cyclostationarity against WSS noise in 4 channel data, at an SNR of -16dB. The results are shown in Figure 1. The time-domain coherence detector results are labeled "GC". The other two curves show the performance of the Ramírez et al. detector and a coherence based modification which uses a STFT, over sampled in time by a factor of 64. It can be seen the oversampling significantly improves the performance of the detector.

These results, along with those presented in [15], support the contention that, in coherence based cyclostationarity detection, it is the correlation lengths that are of paramount importance.

IV-A. A Simulation

This example involves a signal from a multi-channel passive surveillance system with an array of 4 receivers. The received signal $\boldsymbol{x}(n) \in \mathbb{C}^4$, observed by the 4 sensors is modeled as

$$\boldsymbol{x}(n) = \mathbf{a}(\theta)^\dagger \boldsymbol{s}(n) + \boldsymbol{\nu}(n)$$

where $\mathbf{a}(\theta)$ is the steering vector in the source direction θ. The signal $\boldsymbol{s}(n) \in \mathbb{C}^4$ is a QPSK modulated signal with rectangular shaping and a symbol period of 50 ns. It is assumed that the receiver has 133 MHz bandwidth and its center frequency is set at 9.4 GHz. The data is digitally sampled at $8/3 \times 10^8$ samples per second. This gives a cyclic frequency for the QPSK signal of 13.3333 samples/cycle. To apply the detector in [5], the data is resampled by a factor 6/5 with the cyclic frequency becoming 14 samples/cycle. The receiver noise $\boldsymbol{\nu}(n)$ is chosen to be colored WSS noise created by applying a common randomly selected (at each simulation) 20 tap filter to GWN, which is independent across the sensors channels. The results of the simulation are shown in Figure 1 and are discussed above.

V. CONCLUSION

Cyclostationarity, and more generally almost cyclostationarity, are manifest in the mutual coherence of subspaces spanned by sets of time and frequency shifted versions of the random process. Based on this observation, this paper and [15] have explored the construction of both time and spectral domain detectors for the presence cyclostationarity of a given period in WSS noise. However, the theory developed here is not limited to this special case. The ramifications of this observation will be explored in future work.

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