INVARIANCE AND THE BAYESIAN APPROACH TO GENERALIZED COHERENCE TESTS

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ABSTRACT

This paper considers the problem of testing for mutual independence of multiple sets of complex Gaussian vectors. This problem has classical roots in statistics and has been of recent interest in the signal processing literature in connection with multi-channel signal detection. The probability distribution of the maximal invariants under the action of a subgroup of the full invariance group of the problem is derived for both hypotheses. It is shown that for parameter space, the maximal invariants under the action of this subgroup form a compact space on which proper non-informative prior distributions can be constructed. Bayesian likelihood ratios for the maximal invariants are derived for various proper prior distributions. Previously, Bayesian likelihood ratios associated with non-informative prior distributions for this problem could only be constructed through considerably less satisfactory limiting techniques.

***Index Terms*—** Multiple-channel detection, Invariance, Maximal invariant, Wijsman's Theorem

1. INTRODUCTION

Invariances of various types feature prominently in the history of statistical hypothesis testing. In particular, the problem of testing for independence among multiple sets of normal random variables is classical in the statistical literature [1] and is fundamental in modern multi-channel signal detection. This circle of detection problems was studied in the 1970s in connection with in passive sonar applications involving two channels of sensor data (e.g., [2]) and somewhat later with multiple and generalized coherence tests for multiple channels [3, 4, 5]. There has been a recent resurgence of interest in problems of this type, much of it motivated by passive radar applications (e.g., [6, 7, 8, 9, 10]) and spectrum sensing for cognitive radio (e.g., [11, 12, 13]). A space-time perspective was introduced in [14] that generalized this circle of problems and associated statistical methods. Invariances are a recurring theme in this work (e.g., [15, 16, 17, 18, 19, 20, 21]).

The GLR coherence detector was originally derived in [1], although the detector was rediscovered in a number of contexts over the years. This detector takes the form of the determinant of the normalized Gram matrix (described in Section 3) of the multi-channel data vectors. In [22] we gave a Bayesian derivation of this detector in the limiting case of non-informative priors. It turned out to be quite difficult to balance the H_1 and H_0 priors to achieve this. At about the same time Ramírez, et al. [19] used an argument based on the full invariance group of the problem and Wijsman's Theorem [23] to derive the "locally most powerful invariant test" (LMPIT) for this problem for which the test statistic is the Frobenius norm of the normalized Gram matrix of the data. In [21] we considered in some detail the structure of the maximal invariants of the action of the full invariance group on the data space, as well as the maximal invariants of a number of subgroups for the full group. We further derived the probability distributions for these maximal invariants directly.

In this paper we derive a number of Bayesian likelihood ratios for a fairly general case of the type of multi-channel detection problems described above. We use the fact that the maximal invariants for the action of certain subgroups of the full invariance group on the parameter space form a compact space to construct a class of proper invariant priors.

In Section 3, we show that the maximal invariant statistic under the action of the subgroup of block diagonal upper triangular matrices, with positive real diagonals, of the full invariance group, is described equivalently, a set of points on a complex Stiefel manifold, or as a set of semi-unitary matrices or K-frames. Sections 4 and 5 describe the maximal invariants of the parameter space under the action of this subgroup which, forming a compact space, allows the construction of proper invariant non-informative priors in Section 6. Section 7 derives the distributions of the maximal invariant under this subgroup for both hypotheses of the detection problem, and constructs the Bayesian likelihood ratios as integrals. A number of limiting cases are derived.

2. MODEL AND PROBLEM FORMULATION

Consider the following problem. There are M sets, each containing K zero-mean Gaussian random vectors in $\mathbb{C}^N$, $N \geq K$. Within each random vector the components are independently distributed. Arranging the vectors $\{\boldsymbol{x}_{m,1}, \ldots, \boldsymbol{x}_{m,K}\}$ as the columns of the $N \times K$ matrix X_m, the probability density of the m^{th} set of vectors is taken as

$$p(X_m|R_m) = \frac{1}{(2\pi)^{KN} \det{(R_m)}^N} \exp\left(-\frac{1}{2}\text{Tr}\left(X_m R_m^{-1} X_m^\dagger\right)\right)$$

For each m, take the data matrix X_m to belong to the space $\mathcal{X}$ of $N \times K$ matrices with two properties: (i) they have full rank K, and (ii) the $K \times K$ sub-matrix consisting of the first K rows of X is invertible. Note that, with the density assumed distribution, the matrices X_m have these properties with probability one. The reasons for these conditions will become clear in Section 3.

The question to be addressed is whether the M sets of vectors are mutually independent. In order to formulate the associated hypothesis test succinctly, denote the space of $L \times L$ positive definite matrices by $\mathcal{P}_L$, and further denote by $\mathcal{P}_L^M$ the space of $LM \times LM$ block-diagonal positive definite matrices with $L \times L$ blocks. Define a $N \times MK$ matrix $\boldsymbol{X} = (X_1, \ldots, X_M)$ and note that the density

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of $\boldsymbol{X}$ is

$$p(\boldsymbol{X}|\boldsymbol{R}) = \frac{1}{(2\pi)^{KNM}\det(\boldsymbol{R})^N}\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{X}\boldsymbol{R}^{-1}\boldsymbol{X}^\dagger\right)\right)$$
$$= \frac{1}{(2\pi)^{KNM}\det(\boldsymbol{R})^N}\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{R}^{-1}\boldsymbol{X}^\dagger\boldsymbol{X}\right)\right) \quad (1)$$

for $\boldsymbol{R} \in \mathcal{P}_{MK}$. The hypothesis test can now be stated as H_1 against H_0 with

H_1 : $\boldsymbol{X}$ is distributed according (1) for some $\boldsymbol{R} \in \mathcal{P}_{MK}$

H_0 : $\boldsymbol{X}$ is distributed according (1) for some $\boldsymbol{R} \in \mathcal{P}_K^M$.

It is observed in [19] that this detection problem is invariant under the following group actions. First take $\{S_1, \cdots, S_M\}$ to be a set of matrices with each $S_m \in \mathrm{GL}(K,\mathbb{C})$, the group of $K \times K$ invertible complex matrices. In block form $\boldsymbol{S} \in \mathrm{GL}(K,\mathbb{C})^M$ is the invertible block diagonal matrix with blocks $\{S_1, \cdots, S_M\}$. Consider the (right) action of $\mathrm{GL}(K,\mathbb{C})^M$ on $\boldsymbol{X} \in \mathbb{C}^{N\times MK}$ given by

$$\boldsymbol{X} \to \boldsymbol{X}\boldsymbol{S}$$

and the corresponding action of $\mathrm{GL}(K,\mathbb{C})^M$ on $\boldsymbol{R} \in \mathcal{P}_{KM}$,

$$\boldsymbol{R} \to \boldsymbol{S}^\dagger\boldsymbol{R}\boldsymbol{S}. \quad (2)$$

The action (2) preserves the block-diagonal structure of the matrices in $\mathcal{P}_K^M \subset \mathcal{P}_{KM}$. Under the combined group action it is easily seen that

$$p(\boldsymbol{X}\boldsymbol{S}|\boldsymbol{S}^\dagger\boldsymbol{R}\boldsymbol{S})|\boldsymbol{S}|^{2N} = p(\boldsymbol{X}|\boldsymbol{R})$$

if $\boldsymbol{R}$ is structured either as in H_0 or as in H_1. Additionally, the group S_M of permutations that permute the order of the M sets of vectors also defines an invariance of the problem. If P_σ is an $M\times M$ permutation matrix corresponding to the permutation $\sigma \in \mathrm{S}_M$, then its respective actions on $\mathcal{X}^M$ and $\mathcal{P}_{KM}$ are given by

$$\boldsymbol{X} \to \boldsymbol{X}\boldsymbol{P}_\sigma$$
$$\boldsymbol{R} \to \boldsymbol{P}_\sigma^\dagger\boldsymbol{R}\boldsymbol{P}_\sigma$$

where $\boldsymbol{P}_\sigma = P_\sigma \otimes I$, with I the $K \times K$ identity matrix. The full group of invariances is generated by the composition of the above actions of $\mathrm{GL}(K,\mathbb{C})^M$ and S_M and will be denoted by F.

Another symmetry of the problem involves the unitary group $U(N)$ with action on $\mathcal{X}^M$ by

$$\boldsymbol{X} \to \boldsymbol{U}\boldsymbol{X}.$$

This action leaves $p(\boldsymbol{X}|\boldsymbol{R})$ invariant with the identity action on the parameter $\boldsymbol{R}$ and is consequently unimportant in the context of constructing invariant tests. In fact, this invariance is a consequence of the sufficiency of the statistic $\boldsymbol{X}\boldsymbol{X}^\dagger$ for $\boldsymbol{R}$.

The approach taken in [21] was to consider the maximal invariants under the action of successively larger subgroups of the full invariance group.

3. MAXIMAL INVARIANTS IN $\mathcal{X}^M$

The structure of the maximal invariants of the groups action of F on the space $\mathcal{X}^M$ can be understood by decomposing F into subgroups. Begin with the data $X \in \mathcal{X}$ from one of the M sets. Every element $G \in \mathrm{GL}(K,\mathbb{C})$ can be uniquely decomposed as $G = TU$, where T is an upper triangular complex matrix with positive real diagonal elements and $U \in U(K)$ is a unitary matrix. The subgroup of upper triangular complex matrices with positive real diagonal elements will be denoted by $\mathcal{T}_K \subset \mathrm{GL}(K,\mathbb{C})$. The subgroup of $\mathrm{GL}(K,\mathbb{C})^M$ consisting of block diagonal matrices with diagonal blocks in $\mathcal{T}_K$ will be denoted by $\mathcal{T}_K^M$. Applying all of the elements of $\mathcal{T}_K$ to X generates a set of matrices in $\mathcal{X}$, the orbit O_X of X under the action of $\mathcal{T}$; i.e.,

$$\mathcal{O}_X = \{XT | T \in \mathcal{T}_K\}.$$

A matrix $X \in \mathcal{X}$ consists of K linearly independent columns, since it was assumed to have full rank. Now each such matrix can be uniquely decomposed $\tilde{X}T$, where the columns of $\tilde{X}$ form an orthonormal set, i.e., $\tilde{X}^\dagger\tilde{X} = I_K$, and $T \in \mathcal{T}_K$. Thus, each orbit under the action of $\mathcal{T}_K$ on $\mathcal{X}$ is associated uniquely with a K-frame $\tilde{X}$ of K orthonormal vectors in $\mathbb{C}^N$. The space of all K-frames of vectors in $\mathbb{C}^N$ is the Stiefel manifold $V_{VK,N}$, which is a smooth complex manifold. Thus, overall the maximal invariant for the action of $\mathcal{T}_K^M$ on $\mathcal{X}^M$ are the ordered M element sets of K-frames $\tilde{\boldsymbol{X}} = (\tilde{X}_1, \ldots, \tilde{X}_M)$ which is a element of $V_{K,N}^M$.

In [21] we derived the maximal invariants associated with action of F on $\mathcal{X}^m$ and showed that the maximal invariants consist of the unordered sets of M, K-dimensional subspaces $(\langle X_1\rangle, \ldots, \langle X_M\rangle)$ of $\mathbb{C}^N$, where $\langle X_1\rangle$ denotes the subspace spanned by the columns of X_1. In this paper, however, we will be able to achieve our aims by considering only the maximal invariants associated with the group action of the subgroup $\mathcal{T}_K^M$ on $\mathcal{X}^M$.

4. MAXIMAL INVARIANTS IN $\mathcal{P}_{MK}$

This paper takes a Bayesian approach to invariance and coherence detection and consequently it is necessary to determine the maximal invariants for the action of the group F on the parameter space $\mathcal{P}_{MK}$, or at least for the action of certain subgroups of F.

As in the preceding section, begin by considering the group action of the subgroup $\mathcal{T}_K^M$ of F. Recall that action of $\mathcal{T}_K^M$ on $\mathcal{P}_{MK}$ is

$$\boldsymbol{R} \to \boldsymbol{T}^\dagger\boldsymbol{R}\boldsymbol{T}$$

for $\boldsymbol{T} \in \mathcal{T}_K^M$. The transformation of the m^{th} $K \times K$ diagonal block of $\boldsymbol{R}$ under this action is

$$R_m \to T_m^\dagger R_m T_m$$

where $T_m \in \mathcal{T}_K$. For any R_m there is a unique choice of T_m such that $R_m \to I_K$, and so along any orbit in $\mathcal{P}_{MK}$ there is a single point at which the diagonal blocks of $\boldsymbol{R}$ are all the identity matrix I_K. Thus, the maximal invariant for the action of $\mathcal{T}_K^M$ consists of the positive definite matrices with all $K \times K$ diagonal blocks R_m equal to I_K. Denote this space of matrices by $\tilde{\mathcal{P}}_{MK}$. It is important note here that the set of matrices $\mathcal{P}_K^M$ associated with the hypothesis H_0 forms a single orbit under the action of $\mathcal{T}_K^M$, and the value of the maximal invariant for this orbit is I_{MK}, the $MK \times MK$ identity matrix.

The maximal invariants under the action of the full invariance group F on $\mathcal{P}_{MK}$, although interesting, have a complex structure which makes working with them more difficult. Fortunately, the maximal invariants for subgroup $\mathcal{T}_K^M$ are sufficient for our needs.

5. RELATIVELY INVARIANT MEASURES

Background on group actions, multipliers, and Haar measure for this section can be found in [24]. The group $\mathrm{GL}(K,\mathbb{C})$ is a unimodular group, which means that the modular function $\Delta : \mathrm{GL}(K,\mathbb{C}) \to$

$\mathbb{R}^+$ which relates the left and right Haar measures has value 1 on all $\text{GL}(K, \mathbb{C})$. Thus, the left and right Haar measures for this group are equal and are given by

$$d\nu(S) = \frac{dS}{|S|^{2K}}$$

where dS denotes the Lebesgue measure on $\mathbb{C}^{K\times K}$. This simply generalizes to the Haar measure on $\text{GL}(K, \mathbb{C})^M$

$$d\nu(\boldsymbol{S}) = \prod_{m=1}^{M} \frac{dS_m}{|S_m|^{2K}} = \frac{d\boldsymbol{S}}{|\boldsymbol{S}|^{2K}}.$$

The right Haar measure on the subgroup $\mathcal{T}_K$ of $\text{GL}(K, \mathbb{C})$ is

$$d\nu^{(r)}_{\mathcal{T}_K}(T) = \prod_{j=1}^{K} t_{jj}^{-2j+1} dT$$

where dT is the Lebesgue measure on $\mathcal{T}_K$, and so the right Haar measure on $\mathcal{T}_K^M$ is

$$d\nu^{(r)}_{\mathcal{T}_K^M}(\boldsymbol{T}) = \prod_{m=1}^{M} d\nu^{(r)}_{\mathcal{T}_K}(T_m).$$

Under the action of $\text{GL}(K, \mathbb{C})^M$, the Lebesgue measure $d\boldsymbol{X}$ on $\mathcal{X}^M$ has multiplier $\chi_D(\boldsymbol{S}) = |\boldsymbol{S}|^{2N}$, that is, for $\boldsymbol{S} \in \text{GL}(K, \mathbb{C})^M$,

$$d(\boldsymbol{X}\boldsymbol{S}) = \chi_D(\boldsymbol{S}) d\boldsymbol{X} = |\boldsymbol{S}|^{2N} d\boldsymbol{X}.$$

An F-invariant measure on $\mathcal{X}^M$ can be constructed as

$$d\mu_D(\boldsymbol{X}) = \frac{d\boldsymbol{X}}{\prod_{m=1}^{M} |X_m^\dagger X_m|^N}.$$

Note that this measure is also relatively invariant for $\text{GL}(K, \mathbb{C})$, with multiplier $1/\Delta(\boldsymbol{S})$ since $\Delta(\boldsymbol{S}) = 1$ for $\text{GL}(K, \mathbb{C})$. For the decomposition $\boldsymbol{X} = \tilde{\boldsymbol{X}}\boldsymbol{T} = (\tilde{X}_1 T_1, \ldots, \tilde{X}_m T_m)$,

$$d\mu_D(\boldsymbol{X}) = d\nu^{(r)}_{\mathcal{T}_K^M}(\boldsymbol{T}) \times \prod_{m=1}^{M} d\mu_{V_{K,N}}(\tilde{X}_m) \tag{3}$$

where $d\mu_{V_{K,N}}$ denotes the invariant measure on the Stiefel manifold $V_{K,N}$. The measure $\prod_{m=1}^{M} d\mu_{V_{K,N}}(\tilde{X}_m)$ is the invariant measure on the maximal invariant of the group action of $\mathcal{T}_K^M$ on $\mathcal{X}^M$.

Moving to measures on the parameter space $\mathcal{P}_{MK}$, it is seen that under that action of $\boldsymbol{S} \in \text{GL}(K, \mathbb{C})^M$, the Lebesgue measure on $\mathcal{P}_{MK}$ has multiplier (Jacobian) $\chi_P(\boldsymbol{S}) = |\boldsymbol{S}|^{MK}$. Thus, an F invariant measure on $\mathcal{P}_{MK}$ can be constructed as

$$d\mu_P(\boldsymbol{R}) = \frac{d\boldsymbol{R}}{\prod_{m=1}^{M} |R_m|^{MK}}. \tag{4}$$

Writing $\boldsymbol{R} = \boldsymbol{T}^\dagger \tilde{\boldsymbol{R}} \boldsymbol{T}$ in terms of the maximal invariant for the action of $\mathcal{T}_K^M$, it can be shown that

$$d\mu_P(\boldsymbol{R}) = 2^{MK} d\nu^{(r)}_{\mathcal{T}_K^M}(\boldsymbol{T})\, d\tilde{\boldsymbol{R}} \tag{5}$$

where $d\tilde{\boldsymbol{R}}$ is the Lebesgue measure on $\tilde{\mathcal{P}}_{MK}$.

6. PRIOR DISTRIBUTIONS ON THE MAXIMAL INVARIANTS

A natural invariant prior on the parameter space $\mathcal{P}_{MK}$ is the invariant measure (4). However, this meaure gives an improper prior on $\mathcal{P}_{MK}$, which is inconvenient when the goal is to compute Bayesian marginalized likelihoods. In [22], we used a limiting approach to produce a Bayesian likelihood ratio, but balancing the priors for H_1 and H_0 in the limit turned out to be challenging. Our approach here is to use the invariances of the problem to move the choice of priors to some maximal invariant.

In Section 4, we showed that the maximal invariant for the action of $\mathcal{T}_K^M$ on $\mathcal{P}_{MK}$ was the space of matrices $\tilde{\mathcal{P}}_{MK}$ and also that the quotient measure on $\tilde{\mathcal{P}}_{MK}$ was just he Lebesgue measure $d\tilde{\boldsymbol{R}}$. Furthermore, from [20], it can be shown that, for integers $Q \geq MK$,

$$\int_{\tilde{\mathcal{P}}_{MK}} |\tilde{R}|^{Q-MK} d\tilde{R} = \frac{\text{vol}(V_{K,Q})^M}{\text{vol}(V_{MK,Q})}, \tag{6}$$

so that

$$\frac{\text{vol}(U(MK))}{\text{vol}(V_{K,MK})^M} d\tilde{R} \tag{7}$$

is a proper uniform prior probability measure on $\tilde{\mathcal{P}}_{MK}$. Note that for any N, $V_{N,N}$ is just $U(N)$. All the above distributions are invariant under $U(K)^M \times S_M$. As mentioned above, under H_0, the value of the maximal invariant is $\tilde{R} = I_{MK}$ and so no prior is necessary.

7. BAYESIAN LIKELIHOOD RATIOS

Our first task is to compute the probability distribution for the maximal invariant for the action of $\mathcal{T}_K^M$ on $\mathcal{X}^M$. Making the change of variables $\boldsymbol{X} \to \tilde{\boldsymbol{X}}\boldsymbol{T}$ and using (3),

$$\begin{aligned} &p(\tilde{\boldsymbol{X}} | \boldsymbol{R} \in \mathcal{P}_{KM}) \\ &= \frac{1}{\pi^{MKN}} \int_{\mathcal{T}_K^M} \frac{\exp\left(-\text{Tr}(\tilde{\boldsymbol{R}}^{-1} \boldsymbol{T}^\dagger \tilde{\boldsymbol{X}}^\dagger \tilde{\boldsymbol{X}} \boldsymbol{T})\right)}{|\boldsymbol{T}\tilde{\boldsymbol{R}}\boldsymbol{T}^\dagger|^N} d\nu^{(r)}_{\mathcal{T}_K^M}(\boldsymbol{T}) \end{aligned} \tag{8}$$

with respect to invariant measure $\prod_{m=1}^{M} d\mu_{V_{K,N}}(\tilde{X}_m)$ on $V_{K,N}^M$. This only depends on the maximal invariant $\tilde{\boldsymbol{R}}$. Under H_0,

$$p(\tilde{\boldsymbol{X}} | \boldsymbol{R} \in \mathcal{P}_K^M) = \frac{1}{\text{vol}(V_{K,M})^M}$$

that is, it is uniform with respect to the above measure on $V_{K,N}^M$. Now, marginalizing (8) with respect to the prior (6), the likelihood ratio becomes

$$\begin{aligned} \frac{p(\tilde{\boldsymbol{X}}|H_1)}{p(\tilde{\boldsymbol{X}}|H_0)} = 2^{MK}\kappa \int\limits_{\tilde{\mathcal{P}}_{MK}\times\mathcal{T}_K^M} &\frac{\exp\left(-\text{Tr}(\tilde{\boldsymbol{R}}^{-1} \boldsymbol{T}^\dagger \tilde{\boldsymbol{X}}^\dagger \tilde{\boldsymbol{X}} \boldsymbol{T})\right)}{|\boldsymbol{T}\tilde{\boldsymbol{R}}\boldsymbol{T}^\dagger|^N} \\ &\times |\tilde{\boldsymbol{R}}|^{Q-MK} d\nu^{(r)}_{\mathcal{T}_M^K}(\boldsymbol{T}) d\tilde{\boldsymbol{R}} \end{aligned} \tag{9}$$

where the constant κ is

$$\kappa = \frac{\text{vol}(V_{K,N})^M \text{vol}(V_{MK,Q})}{2^{MK}\pi^{MKN} \text{vol}(V_{K,Q})^M}.$$

Using (5), this can be rewritten as

$$\frac{p(\tilde{\boldsymbol{X}}|H_1)}{p(\tilde{\boldsymbol{X}}|H_0)} = \kappa \int_{\mathcal{P}_{MK}} |\boldsymbol{R}^{-1}|^{N+MK-Q} \exp\left(-\mathrm{Tr}(\boldsymbol{R}^{-1}\tilde{\boldsymbol{X}}^\dagger \tilde{\boldsymbol{X}})\right) |\boldsymbol{R}_D^{-1}|^Q d\boldsymbol{R} \tag{10}$$

where $\boldsymbol{R}_D$ is the block diagonal matrix whose $K \times K$ diagonal blocks equal those of $\boldsymbol{R}$. For $Q = MK$ this this the Bayesian likelihood ratio associated with the uniform (non-informative) prior on $\tilde{\mathcal{P}}_{MK}$. At the current time we have no way of computing the integral in the general case and it remains on open research problem. We can however compute a number of special limiting cases.

The first case is an asymptotic approximation for large N and an arbitrary but fixed value of Q. Rewrite the integral in (10) as

$$\mathcal{I}_N = \int_{\mathcal{P}_{MK}} |\boldsymbol{R}_D^{-1/2}\boldsymbol{R}\boldsymbol{R}_D^{-1/2}|^Q |\boldsymbol{R}^{-1}|^{N+MK} e^{-\mathrm{Tr}(\boldsymbol{R}^{-1}\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}})} d\boldsymbol{R}$$

and note that the term $|\boldsymbol{R}^{-1}|^{N+MK} \exp\left(-\mathrm{Tr}(\boldsymbol{R}^{-1}\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}})\right)$ is an un-normalized complex inverse Wishart density. Its maximum occurs at $\boldsymbol{R}_{\max} = \tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}/(N+MK)$ and it becomes progressively more highly peaked about this value as $N \to \infty$. In the limit, the term $|\boldsymbol{R}_D^{-1/2}\boldsymbol{R}\boldsymbol{R}_D^{-1/2}|^Q$ can be replaced by $|\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}/(N+MK)|^Q$ to give

$$\mathcal{I}_N \sim |\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}/(N+MK)|^Q \int_{\mathcal{P}_{MK}} |\boldsymbol{R}^{-1}|^{N+MK} e^{-\mathrm{Tr}(\boldsymbol{R}^{-1}\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}})} d\boldsymbol{R}.$$

Thus,

$$\frac{p(\tilde{\boldsymbol{X}}|H_1)}{p(\tilde{\boldsymbol{X}}|H_0)} \sim \frac{C}{|\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}|^{N-Q}} \quad \text{as } N \to \infty$$

where

$$C = \frac{1}{\pi^{MKN}} \frac{\mathrm{vol}(V_{K,N})^M \mathrm{vol}(V_{MK,Q})}{\mathrm{vol}(V_{K,Q})^M \mathrm{vol}(V_{MK,N})} (N+MK)^{MKQ}.$$

Another special case that can be computed is the large Q limit. This corresponds to prior information that $\tilde{\boldsymbol{R}}$ is concentrated near $\boldsymbol{I}_{MK}$; i.e., H_1 and H_0 are very close. This case is related to the "Locally Most Powerful Invariant Test" (LMPIT) considered in [19] from a non-Bayesian perspective. There they find that the LMPIT statistic is the Frobenius norm of $\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}$. The computation of the large Q limit of the Bayesian likelihood ratio, which uses (9) as the starting point, is quite involved, so here we will just give the result:

$$\frac{p(\tilde{\boldsymbol{X}}|H_1)}{p(\tilde{\boldsymbol{X}}|H_0)} \sim 1 + \frac{N^2\|\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}\|_F^2 - D}{Q}$$

where

$$D = MK\left((M-1)K(N-MK-1) + N^2\right)$$

and $\| \quad \|_F$ denotes the Frobenius norm.

Finally, note that both these limiting cases, along with GLR and LMPIT detectors, are function only of the eigenvalues of $\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}$. This is generally not true of the detectors defined by the likelihood ratios (10) which depend more generally on $\tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{X}}$, and so in a sense the GRL and LMPIT detectors display more symmetry than the detection problem itself.

8. CONCLUSION

The problem of testing for mutual independence of multiple sets of complex Gaussian vectors has been considered from a Bayesian perspective. It was shown that the maximal invariants for the action of the subgroup of upper triangular matrices on the parameter space of positive definite matrices forms a compact space on which proper non-informative prior distributions can be constructed. Bayesian likelihood ratios for the maximal invariants in $\mathcal{X}^M$ were derived for various proper prior distributions. Further work will involve investigating methods for computing the integral (10), such as deriving full asymptotic expansions rather than just the lowest-order terms.

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