Bounds on Multiple Sensor Fusion

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We consider the problem of fusing measurements in a sensor network, where the sensing regions overlap and data are non-negative real numbers, possibly resulting from a count of indistinguishable discrete entities. Because of overlaps, it is, in general, impossible to fuse this information to arrive at an accurate value of the overall amount or count of material present in the union of the sensing regions. Here we study the computation of the range of overall values consistent with the data, and provide a number of results. Posed as a linear programming problem, this leads to questions associated with the geometry of the sensor regions, specifically, the arrangement of their non-empty intersections. We define a computational tool called the fusion polytope, based on the geometry of the sensing regions. Its properties are explored and, in particular, a topological necessary and sufficient condition for this to be in the positive orthant, a property that considerably simplifies calculations, is provided. We show that, in two dimensions, inflated tiling schemes based on rectangular regions fail to satisfy this condition, whereas inflated tiling schemes based on hexagons do.

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1. INTRODUCTION

Examples abound in sensing of measurement processes which, rather than identifying objects or events, merely count them or measure their size, for instance, by integrating a total response over all objects accessible to each individual sensor. One such example arises in people counting algorithms in video-analytics that rely on some optical flow from a moving group rather than separate identification of each

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individual. More specifically, for public safety or business intelligence gathering [Loy et al. 2013], a camera network system with overlapping field of views [Li et al. 2012] is used to count the number of people in an open area, for example, a public square or a shopping mall. Through optical flow method [Benabbas et al. 2010], each camera reports the number of people detected. However, the system is not able to identify which observations from different cameras show the same person, especially in crowded scenes. Another example is measurement of radio activity using Geiger counters. Geiger counters to measure alpha or beta radiation, for instance, have a very limited range (a few centimetres in the case of alpha radiation) and so a distributed array of these would used to measure over a larger area. While these counters measure individual and distinct particles, those measurements are taken as an indication of a nearby radioactive source. Different elements of the array will intercept particles from the same source, leading to multiple counting of radiation sources. Measurement of the total amount of radioactive material in the vicinity of the array suffers the problem of multiple counting, addressed here. Other examples include cell counting techniques, and counts of numbers of Radio frequency (RF) transmitters using overall signal strength (For saving energy, a transmitter only emits "pulse" without identity card information). We assume a network of such sensors each reporting the amount of material within its sensing region. An unachievable aim is to find the total amount of material across the entire sensing domain, as the sensing regions will typically overlap. We focus on finding the range of possible values for the amount of material in the sensing domain consistent with the measurements. In effect this paper attempts to address the two problems of estimation of the amount of material (the case when measurements are non-negative real numbers) and the counting problem (when the measurements are non-negative integers). We shall refer to the latter problem as the *discrete case*.

At a more abstract level, it is not necessary to differentiate sensors by their physical location but possibly by other properties. Envisaged is a situation involving multiple sensors where each sensor is able to measure a different range of properties (such a property might be an amount of some substance in a given spatial region but can be more general), and where the ranges of properties involved are not mutually exclusive. Our initial interest was in the counting of spatially distributed targets, and in this case the property is that of being in a given "sensor region" described in terms of its geographical spread. The methods discussed here apply equally well to measurements where the outcome is a real number, provided only that the quantities being measured are non-negative, and to where the distinguishing properties of the various sensors might be characteristics other than physical location.

As the results and ideas of this paper are very generic, we present a simple and, in some ways, archetypal example. This involves sensors on the ground capable of counting all objects ("targets") close to them in some sense. More specifically, we could consider counts of animals fitted with a small transmitter carrying no identification information, by a network of RF receivers. Each sensor is associated with a "sensor region" within which any target present is detected, without being identified, and forms part of the count for that sensor. In this context we assume a deterministic sensor. It detects if the target is in the sensing region and does not detect if the target is outside the sensing region. As already indicated, the particular property of the objects being detected is that they belong to this sensor region. In this simple case, the regions may be regarded as subsets of $\mathbf{R}^2$. A more complex example might define the sensor region as encompassing all transmissions that are both close to a sensor in $\mathbf{R}^2$ and emit in a certain frequency band; the regions in this case are subsets of $\mathbf{R}^3$. In yet more complex situations, targets may be distinguished by their positions in space and by a number of other features such as colour, emission frequency or energy, or rapidity of movement (in the case of radars measuring Doppler, for example). The targets, then, can be regarded as points in a multi-dimensional space $\mathbf{R}^n$ and each sensor as defining a region of that space over which it is able to detect and count targets or measure the total integrated value of some response over that region.

To formulate the archetypal problem mathematically, we envisage a collection of points ("targets") $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_R$ in $\mathbf{R}^n$ and a collection of "sensor regions" $S_1, S_2, \ldots, S_R$. At this stage, we impose no structure on the sensor regions, other than that they are non-pathological subsets of $\mathbf{R}^n$. Later we will be more precise.

Assumed known is only a little of the geometry; specifically, the overlap regions $S_{r_1} \cap S_{r_2} \cap \cdots \cap S_{r_T}$ for any set of T distinct integers in the range $1 \leq r_t \leq R$ are known to be empty or non-empty. It needs to be stressed that no further information about the overlaps is known; in particular, it is not known how many targets are in these overlaps. To be slightly more specific, it is assumed that we know whether or not an overlap is capable of containing a target of interest, and only that. This condition has to be specified a little more precisely for some of our results in Section 5.

Each sensor reports its measurement to a central processor; thus sensor S_r reports n_r “targets”. The question at hand is “how many targets are there altogether?”; that is, how can we calculate the overall value from these sensor reports? Of course, as Gandhi et al. [2008] note (Theorem 1), it is trivial to see that there is no unique answer to that question, since we do not know how many targets are in overlaps. The simple example of two overlapping sensor regions with say 5 targets reported by sensor S_1 and 7 targets reported by sensor S_2 may have an overall target count of any number between 7 and 12 targets according to how many are in the overlap region $S_1 \cap S_2$. In more generality, the inclusion-exclusion principle provides an answer to the overall count *provided we know how many targets are in intersections of sensor regions*:

$$|\bigcup_{r=1}^{R} S_r| = \sum_{r=1}^{R} |S_r| - \sum_{r_1 \neq r_2} |S_{r_1} \cap S_{r_2}| + \sum_{r_1 \neq r_2 \neq r_3} |S_{r_1} \cap S_{r_2} \cap S_{r_3}| - \cdots + (-1)^{R-1} |S_1 \cap S_2 \cap \cdots S_R|. \quad (1)$$

As already notes, since the information about the number of targets in intersections is typically unavailable, we cannot use this formula to calculate the overall number of targets. It would be possible to set up our problem in terms of a *fusion function* f, that ascribed a value $f(A) \geq 0$ to a sensing region. The required property to make the theory to be described here go through is that of *additivity*:

$$f(S_1 \cup S_2) = f(S_1) + f(S_2),$$

for any pair S_1, S_2 of disjoint regions. This leads, with not too much difficulty, to the inclusion-exclusion principle (1) with $f(S)$ replacing $|S|$.

One might argue then that sensors should be chosen so that these sensor regions do not overlap. As stated in Gandhi et al. [2008], this is impractical for several reasons. Sensor regions do not come in shapes that permit tiling of Euclidean space or even the subset where targets might reside. Since we wish to count all targets, every target should be in at least one sensor region. In the situation where targets might be missed by an individual sensor, it makes sense to have more and larger overlap rather than less.

We emphasize that in this work there is no uncertainty in the model. Each sensor measures the amount of material in its region or counts the number of objects without error. A later paper will example issues around uncertainty for this kind of situation.

The problem to be faced is to find the range of possible values for the number of targets from the information available: the sensor geometry and the target count reports from individual sensors. This paper focuses largely on the problem of ascertaining the minimum number of targets consistent with the target measurements (the maximum number is easily calculated under our assumptions). It is relatively straightforward to see that the set of possible overall values is an interval, as we shall show.

This paper makes use of several mathematical structures. In particular, we use some basic ideas from simplicial complexes and their geometry. These structures are discussed in Zou and Hancock [2010], where they are used in modelling faces. We will also use some linear programming techniques. Again our demands in this regard are fairly basic as can be found in Matousek and Gärtner [2007]. Finally, we will use some results from graph theory and from the topology of the plane.

1.1. Contributions

The key results of this investigation are as follows:

- Formulation of the problem of finding the range of potential values of the overall amount (or count) of material;
- Description of the problem in terms of a simplicial complex (the *fusion complex*), and a precise definition of when this description is exact;

— A linear programming description of the problem and definition of a measurement independent geometrical object, *the fusion polytope*, that enables computation of the range of values;
— Study of the fusion polytope for special cases, for instance, when the fusion complex is a graph, and a relationship to deep problems and results in graph theory;
— A theorem providing a surprising limitation of the fusion complex description in two dimensions, and some preliminary work on hexagonal sensor regions in two dimensions;
— Theorems enabling upper and lower bounds on the range of values of the overall count in terms of subcomplexes and properties of the fusion complex.

1.2. Related Work

Much work has been done on fusion of counts and amount estimation in sensor networks. A comprehensive survey of the discrete case (target counting) in this context is given in Wu et al. [2014]. There, the target counting problem is classified into four major categories, binary counting, numeric counting, energy counting, and compressive counting, depending on the mode of sensing. Within this scheme, our approach fits into the category of numeric counting. Methods discussed in that paper for numeric counting require significant geometric or distributional information (in a statistical sense). The aim of those methods is to provide an estimate of the actual target totals in the sensing domain. One such specific statistical approach is given in Wu et al. [2012]. Regression techniques are used to comprehend the distribution of targets, then a sampling procedure is applied to select subsets of sensors in which any two sensors have no overlapped sensing regions. Guo et al. [2010] apply another statistical approach involving a Poisson assumption for the target distribution across the sensing domain. Some authors, see for instance Zhang et al. [2011], have applied sparsity assumptions and a compressive sensing based approach. In all of these cases an estimate is provided on the actual amount. An idiosyncratic paper by Baryshnikov et al. [2008] uses target-dependent noise to overcome the overlap problem in an elegant way.

Our basic model, at least in the discrete case, is quite similar to that in Gandhi et al. [2008]. In *loc. cit.* each sensor measurement is an integer value, representing the number of distinct targets in its sensing range, and no other distinguishing information about the targets is available. As stated there, this abstraction of a sensor is a fairly good first-order approximation of to a low-cost radar sensor (or infrared or acoustic sensor) able to detect the presence of multiple targets without individual localization. In Gandhi et al. [2008], a simple algorithm is described for providing an upper and lower bound for the number of targets, but this is only effective in one dimensional sensing domains, though they do provide an approximate count for two-dimensional situations under strict geometrical constraints. In a sense this paper of Guo et al. [2010] may be seen as a precursor to ours in that it seeks to provide bounds on the number of targets and realizes that this is a function of the topology of overlaps.

Another application area for the ideas and results presented here is the assessment of the probability that at least one sensor of several will see a given event. This kind of analysis is required if, for instance, we are interested in obtaining a measure of performance for the entire network of sensors: “What is the probability that at least one sensor will detect a target in the observed region?” For such a question, each individual sensor has a known probability of observing the event T, say, and there are possibly unknown probabilities of combinations of multiple sensors observing the event. Problems of this kind are considered in Newell and Akkaya [2009]. An interesting variant on this problem is explored in depth in Mittal and Davis [2004], Lazos and Poovendran [2006], Wu et al. [2007] and Lazos et al. [2007], where detection of targets moving through an area (two or three dimensional) is studied. In *loc. cit.*, sensors, each with their own sensing region, are spread across the area of interest. Targets move through the area in a linear motion and are detected with a designated probability as they cross the region of a given sensor. Since they cross multiple regions they may be detected more than once. The overall probability of detection is required.

A key ingredient in some of the papers mentioned above is the *inclusion-exclusion principle* (see (1)). Connolly [1985] and Liang [2007] exploit this principle to calculate the volumes of protein molecules using NMR techniques. There again the ideas of this paper can be used to provide a crude assessment of the volume while significantly reducing the number of measurements. Indeed our techniques apply wherever the inclusion-exclusion principle could be used *if information about the counts for all intersections of regions were available* but, instead, we only have the information about the non-emptiness of overlaps of the sensor regions is available.

It is important to emphasize that, while formulated in terms of counting targets, the same ideas apply to all of the problems mentioned above, including the case of assessing probability of detection. Each target is assumed to belong to the sensing domain; that is, $\bigcup_{r=1}^{R} S_r$ of the sensor regions. Whether or not the sensors cover the region of interest is an issue not discussed in this paper. Coverage problems of this type have been considered by Ghrist and Baryshnikov [2008]. We finesse this issue by assuming that the region of observation is covered by at least one sensor. Of concern to us is that the sensor regions may overlap so that a given target might be counted several times. As we have said, this problem, or variants of it, are discussed in Guo et al. [2008], Huang and Lu [2007], Fang et al. [2002], Aeron et al. [2006], Prasad et al. [1991] and many other papers.

The remainder of this paper is organized as follows. After a precise formulation of the problem in the first part of Section 2 and some preliminary definitions, we discuss how the problem can be re-formulated in terms of the fusion complex in Section 2.1. It turns out that this formulations connects to deep combinatorial problems, some of which have been addressed (see Grötschel and Lovász [1995] and Schrijver [1995]), in special cases.

We will provide a description of this "range of values" problem in terms of the fusion complexes and linear programming; the latter in Section 3. Reframing this linear programming problem in the dual rather than primal domain enables us to describe, as a computational tool, a polytope (the "fusion polytope"), dependent only on the geometry of the sensor region and, most importantly, independent of the measured data.

In general the fusion polytope does not lie in the positive orthant, but when it does lower bounds for the fused information become much easier. Necessary and sufficient conditions for the fusion polytope to lie in the positive orthant in terms of the geometry of the sensor regions are given. Computation of the fusion polytope is non-trivial in general, as illustrated by the simple case when the fusion simplicial complex is a graph in Section 4. For planar regions, we will provide, in Section 5, a description of some interesting sensor configurations that correspond to positivity of the fusion polytope. Then, in Section 6, we will give some descriptions of the fusion complex for sensor regions in $\mathbf{R}^n$. The procedure of calculation of the minimum number of targets will be summarized in Section 7.

The theory will be illustrated with examples of simple cases in which a description of the extreme points of the fusion polytope is possible. The sensor configurations for which the simplicial complex is a graph are discussed in some detail.

2. PROBLEM FORMULATION

Our aim is to discuss the problem of counting of targets using a sensor network. We recall that this is a surrogate for a large collection of problems where target response is integrated over a sensor region, or where the target might be the amount of some material being sensed and so be non-negative real valued rather than non-negative integer valued. Fusion of this data is required to provide an overall total of targets or material.

Assume a collection of R sensors, a *sensor configuration* $\mathbf{S}$, labelled by the regions they observe $S_1, S_2, \ldots, S_R$, all subsets of $\mathbf{R}^n$. While this simple definition will suffice for much of this paper, later we will need to be more rigorous.

We assume the following properties of the collection of sensor regions:

Coverage. That the union of the sensor regions $\bigcup_{r=1}^{R} S_r$ is the entire sensing domain Ω. *Nothing escapes detection.*

Irredundancy. That there is no redundancy of sensors; that is, no sensor region is entirely contained in the union of the others.

While coverage is a fairly natural assumption; after all we are surely only interested in the region that can be sensed, irredundancy is less clear. In fact, it may well be unacceptable in some applications. Nonetheless, it simplifies calculations and is not too unreasonable.

At this stage, we remark that there are currently no topological assumptions such as openness, closedness, or connectedness, on the sensor regions; they are merely sets with all of the potential pathology that entails. Later in the paper some topological restrictions will be needed.

Measurements made by the sensors in the network are collected into a *sensor measurement vector* $\mathbf{n} = (n_1, n_2, \dots, n_R)$.

Definition 2.1. An *atom* is a *non-empty* set of the form

$$\langle i_1, i_2, \dots, i_T \rangle = S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_T} \cap S_{j_1}^c \cap S_{j_2}^c \cap \cdots \cap S_{j_S}^c, \tag{2}$$

where ^{c} denotes set complement (with respect to the sensing domain Ω), and where $\{i_1, i_2, \dots, i_T, j_1, j_2, \dots, j_S\}$ is an enumeration of the integers from 1 to R (without repetition of course). The set of all atoms is denoted by $\mathfrak{A}(\mathbf{S})$.

We recall that a Boolean algebra of sets is a collection of subsets of some domain, closed under finite unions, intersections, and complements. Atoms are minimal *non-empty* elements of the Boolean algebra generated by the sensor regions $S_1, S_2, \dots, S_R$, *qua* subsets of the sensing domain. Note that, for any atom, the number T in (2) has to be positive, since the intersection of all S_m^c is empty because of the coverage assumption. The atoms form a partition of the sensing domain. The *specification* of a sensor configuration $\mathbf{S}$ is really a statement of which intersections of the form (2) are non-empty, and which sensor regions these non-empty intersections are contained in. This specifies the entire geometrical requirements of the problem.

While the atom structure is necessary to develop the theory, it does not need to be computed for actual calculation of the range of measurements. All that is needed for that calculation is knowledge of whether or not sensor regions intersect as given the specification.

There is one further constraint that will require consideration and indeed will play a significant role in the later sections.

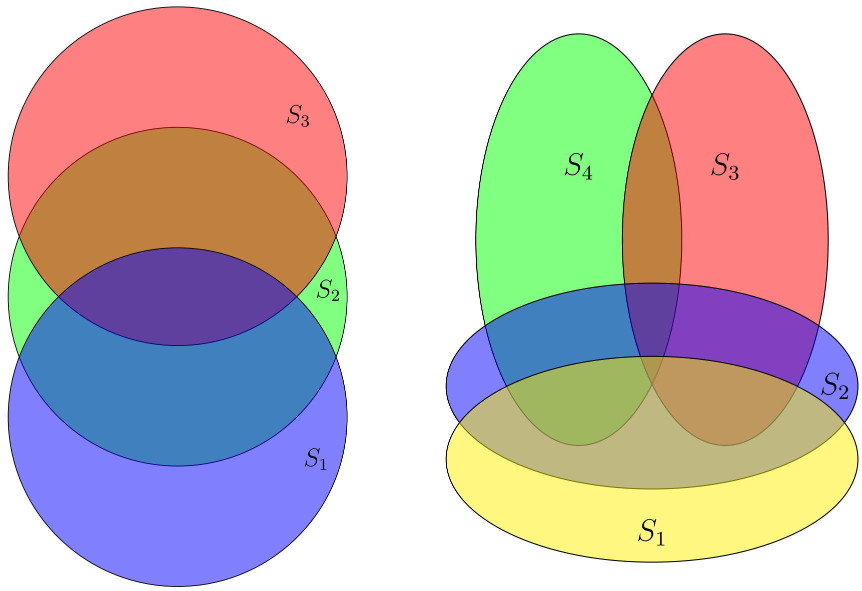
Definition 2.2. Given a sensor configuration $\mathbf{S} = (S_1, S_2, \dots, S_R)$, if no non-empty intersection of sensor regions $\langle i_1, i_2, \dots, i_R \rangle$ is entirely contained in the union of different sensor regions $\cup S_j$ where $j \notin \{i_1, i_2, \dots, i_N\}$ then the sensor configuration is said to be *generic*; otherwise the configuration is *degenerate*.

The problem of finding the minimal bounds differs significantly in the degenerate case from the generic case. Indeed the linear programming formulation is considerably simpler for the generic case. Unfortunately, there are many reasonable situations that are not generic. Examples (the simplest irredundant and a slightly more complicated one) where the generic condition fails are given in Fig. 1 but, as described later, much more natural sensor configurations can fail to be generic.

To be clear, coverage and irredundancy are assumed throughout the paper; assumption of genericity is always stated explicitly. Having defined the basic structures, we reiterate the problem: given a sensor configuration $\mathbf{S}$ and a sensor measurement vector $\mathbf{n}$ we aim to find the range of possible overall values consistent with $\mathbf{S}$ and $\mathbf{n}$.

2.1. Simplicial Complex Formulation

A sensor configuration can be described in terms of a geometrical object called a *simplicial complex*. For a treatment of these structures, we refer the reader to Cheng and Tan [2004] and Liu et al. [2008]. A simplicial complex, in our context, is a collection of subsets Σ of the set $\{1, 2, \dots, R\}$, where R is the number of sensors in the network, with the property that if $\sigma \in \Sigma$ and $\tau \subset \sigma$ then $\tau \in \Sigma$. All singletons $\{r\}$ $(r = 1, 2, \dots, R)$ are assumed to be in Σ, as is the empty set $\emptyset$. The *dimension* of a *simplex* $\sigma \in \Sigma$ is just $\dim \sigma = |\sigma| - 1$, where $|A|$ is the number of elements of the set A. It is useful to think of points of $\{1, 2, \dots, R\}$ (that is singletons) as *vertices*, simplices σ of dimension 1 as edges joining the elements



(a) Simplest degenerate case: $S_1 \cap S_3 \subset S_2$. (b) More complicated example: $S_3 \cap S_4 \cap S_1 \subset S_2$.

Fig. 1: Simple cases of degeneracy.

of the set σ, simplices σ of dimension 2 as triangles with vertices the elements of σ, and so on. The resulting geometrical object is called a *geometrical realization* of the simplicial complex.

The *dimension* of a simplicial complex is the maximum of the dimensions of all of its simplices. The subsets of a simplex σ of dimension $\dim \sigma - 1$ are called the *faces* of σ.

This abstract simplicial complex always has a geometrical representation. The *Geometric Realization Theorem* [Alexandroff and Hopf 1935] states that a simplicial complex of dimension d has a geometric representation in $\mathbf{R}^{2d+1}$, where abstract simplices are represented by geometrical ones and where the abstract concept of face corresponds to the geometrical faces of a simplex. The geometrical picture is a very useful device in visualising the structure of the problem we have described in Section 2, and solutions to some special cases.

A given sensor configuration $\mathbf{S} = \{S_1, S_2, \ldots, S_R\}$ then maps to a simplicial complex, termed the *nerve* of the configuration as follows. The *nerve* (cf. [Alexandroff and Hopf 1935]) of $\mathbf{S}$ is the simplicial complex $\Sigma(\mathbf{S})$ whose vertices are the numbers $\{1, 2, \ldots, R\}$ and where $\sigma \in \Sigma$ if $\bigcap_{r \in \sigma} S_r \neq \emptyset$. It is straightforward to check that this is indeed a simplicial complex. We call this the *fusion complex*.

The ability of the simplicial complex to represent the intersection structure faithfully breaks down when the sensor configuration is degenerate, as in Fig. 1. In the case of Fig. 1(a), the nerve consists of all subsimplices of a triangle, but as can be seen there is no atom that represents one of the 1-simplices. In effect $S_1 \cap S_2 \cap S_3 = S_1 \cap S_3$ so that in the simplicial complex picture two different simplices correspond to the same atom. In similar vein, the list of simplices corresponding to atoms in Fig. 1(b), is

$$\{1,2,3,4\},\ \{1,2,3\},\ \{1,2,4\},\ \{2,3,4\},\ \{1,2\},\ \{2,3\},\ \{2,4\},\ \{3,4\},\ \{1\},\ \{2\},\ \{3\},\ \{4\}, \emptyset,$$

which is not the nerve of this set of regions, since it does not contain the simplex $\{1, 3, 4\}$. These examples are characteristic, as the following simple theorem shows.

THEOREM 2.3. *Given a sensor configuration* $\mathbf{S}$*, the assignment of an atom* $\langle i_1, i_2, \ldots, i_t\rangle$ *to the simplex* $\sigma = \{i_1, \ldots, i_t\}$ *is* $1-1$ *from the atoms* $\mathfrak{A}(\mathbf{S})$ *of* $\mathbf{S}$ *to (the simplices of)* $\Sigma(\mathbf{S})$ *if and only if the sensor configuration is generic.*

PROOF. The assignment of

(1) a non-empty atom given by $S_{i_1} \cap \ldots \cap S_{i_t}$ to the simplex $\sigma = \{i_1, \ldots, i_t\}$ and
(2) the empty intersection to $\emptyset$

is a bijection of sets as long as the intersections $S_{i_1} \cap \ldots \cap S_{i_t}$ are non-empty and distinct with the single exception of the empty set. This is exactly the definition of generic. □

3. CALCULATION OF THE RANGE OF OVERALL VALUES

We suppose a sensor configuration $\mathbf{S} = \{S_1, S_2, \ldots, S_R\}$ and sensor measurements $n_1, n_2, \ldots, n_R$. For each atom $\sigma = \langle i_1, i_2, \ldots, i_r\rangle$, assume that the amount of material (count) in that intersection is m_σ. Calculation of all possible overall amounts of material in the sensing domain, consistent with the reports from each sensor region is required. Since atoms are disjoint, this is $\sum_{\sigma \text{ atom}} m_\sigma$. An upper bound is the sum of the sensor measurements $\sum_{r=1}^{R} n_r$. It is essentially trivial to see that, if the collection of sensor regions is irredundant, this upper bound is indeed the maximum possible (the *max-bound*), achieved by putting all of the value in a sensor region S_r into $S_r \backslash \bigcup_{r' \neq r} S_{r'}$.

We also need to find the minimum consistent with the measurements, which we term the *min-bound*. Its calculation is formulated in terms of a linear programming problem. The variables in the *primal* linear programming problem are the values m_σ in each atom, σ. The constraints in this case are that the values in the atoms that are subsets of any given sensor region have to sum to the sensor measurement for that sensor region, since any such region is a disjoint union of its atoms minimize the sum of all of the atom values m_σ. Formally, the constraints are

$$\sum_{\substack{a \subset S_r \\ a \text{ is an atom}}} m_a = n_r \ (r = 1, 2, \ldots, R) \tag{3}$$

and of course that $m_a \geq 0$ for all atoms a. To obtain the min-bound, we seek the minimum value of $\sum_{a \text{ atom}} m_a$, subject to the constraints in (3). More succinctly, we let $A = A_{\mathbb{S}}$, with the subscript dropped if the meaning is clear, be the matrix corresponding to the linear equations given in (3); that is, the σ, r entry of $A_{\mathbb{S}}$ is 1 if the atom σ is in the sensor region S_r and 0 otherwise. The sensor measurement vector is $\mathbf{n} = (n_1, n_2, \ldots, n_R)$ and the atom value vector is $\mathbf{m} = (m_a)$. Then the linear programming formulation gives that min-bound for this sensor configuration $\mathbf{S}$ and sensor measurement vector $\mathbf{n}$ is

$$\mathfrak{m}(\mathbf{S}, \mathbf{n}) = \min\{\mathbf{1} \cdot \mathbf{m} : A\mathbf{m} = \mathbf{n}, \quad \mathbf{m} \geq 0\}, \tag{4}$$

where $\cdot$ indicates the dot product of the two vectors, so that $\mathbf{1}.\mathbf{m}$ is the sum of the entries in $\mathbf{m}$.

The following theorem then addresses the range of values.

THEOREM 3.1. *Given a sensor configuration* $\mathbf{S} = (S_1, S_2, \ldots, S_R)$ *and sensor measurements* $\mathbf{n} = (n_1, n_2, \ldots, n_R)$*, every (real) value between the max-bound and the min-bound is achievable.*

PROOF. The function

$$f(\mathbf{m}) = \sum_{a \text{ atom}} m_a$$

is continuous as well as defined on a bounded, convex polytope which is compact, and connected. This function $f(\mathbf{m})$ then satisfies the intermediate value theorem. □

The (discrete target) integer case is more difficult and interesting. The following theorem begins to address the problem.

THEOREM 3.2. *If* $\mathbf{S}$ *is generic, then for every set of sensor measurements that are integers, every integer value between the minimum overall (integer) count and max-bound is achievable by integer assignments to atoms.*

PROOF. It is enough to observe that if $\mathbf{n}$ is the sensor measurement vector and k is a possible achievable integer value for the overall count, then so is $k+1$ unless k is the maximum possible count. Suppose first that there is no target in an intersection $S_i \cap S_j$ with $i \neq j$. Then k is the maximum possible count. Otherwise, suppose that a target t is in $S_i \cap S_{j_1} \cap \cdots \cap S_{j_r}$, and in no other sensor regions.

This target is moved into $\left(S_{j_1} \cap S_{j_2} \cap \cdots \cap S_{j_r}\right) \backslash S_i$, which is possible because of genericity. Then insert another target in $S_i \backslash \bigcup_{i=1}^{r} S_{j_i}$. This keeps the sensor measurement for every sensor region unchanged but increases the overall value by 1. Moreover, it is clear that integers are assigned to atoms by this proof. $\square$

We note that the generic property is sufficient but not necessary for the conclusion of Theorem 3.2. Straightforwardly, the first configuration in Fig. 1 can be seen to fail to be generic but still have the property that the range of integer overall values forms an interval for any sensor measurement vector. On the other hand, consider the sensor configuration in Fig. 2. If each sensor region has a count of 1,

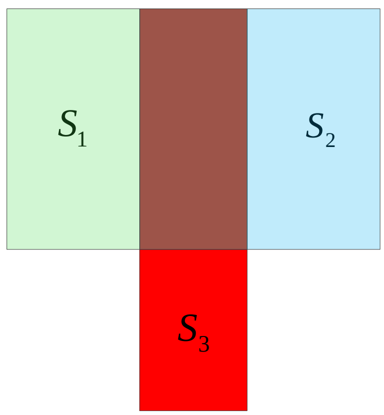


Fig. 2: Example where the integer values do not form an interval.

then 3 is clearly the maximum consistent count and 1 is the minimum, but there is no achievable overall count of 2 with integer sensor measurements.

For the integer case, Theorem 3.2 is not quite as good as we would like, however. While the maximum overall count associated with integer measurements is the same as the max-bound, since this is just the sum of the individual sensor counts, this is clearly not the case for the min-bound and its integer counterpart. Ideally, we would like that the min-bound should be achievable by integer atom counts when the measurements are integers. To see that this fails, consider the case of three sensor regions S_1, S_2, S_3 with only pairwise non-empty intersections: $S_i \cap S_j \neq \emptyset$ for all $i, j = 1, 2, 3$. As we shall see later, the min-bound is $\frac{n_1+n_2+n_3}{2}$ where n_i is the count in region S_i. This is not always an integer and so cannot be achieved by integer atom counts. One might ask whether the smallest integer greater than or equal to the min-bound is achievable by integer atom counts. We have been unable to definitively answer this question, though our experiments suggest that it is always true for generic sensor configurations.

In principle (4) provides a mechanism for calculation of the min-bound, and therefore by Theorem 3.1 the entire range of possible values but, for large numbers of sensors, it becomes impractical. Moreover it suffers the problem that, whenever the measurements change, an entirely new linear programming (or integer linear programming) calculation is needed. What is required is a formulation of the problem that provides a simple machine to go from sensor measurements to min-bounds. This machine itself should be dependent only on the geometry of the sensor regions, with only the inputs of sensor measurements changing. In other words, we would prefer a computable formula into which insertion of the sensor measurements presents the value of (4).

Progress towards such a solution, is obtained via the dual linear programming formulation. Since the primal formulation in (4) is in standard form, the dual problem is (see Dantzig [1998], p.128) expressed in terms of dual variables $\mathbf{y} = (y_r)$ indexed by sensor regions and states

$$\max\{\mathbf{n} \cdot \mathbf{y} : A^T\mathbf{y} \leq \mathbf{1}\}. \tag{5}$$

The key difference between this and the usual dual-primal formulation (see Dantzig [1998]) is that, because the primal is stated in terms of equalities, the dual variables have no restrictions other than the linear inequality $A^T\mathbf{y} \leq \mathbf{1}$; in particular, $\mathbf{y}$ is not required to be non-negative. As a result the feasible region (the convex set described by the constraints) is not, in general, bounded. Existence of solutions therefore becomes an issue.

As an illustration of the problems that can arise consider the case given in Fig. 1(a). The matrix A is obtained as

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{6}$$

and the dual constraint becomes

$$A^T\mathbf{y} \leq 1. \tag{7}$$

This has a solution $(1, 1, -1)$ which gives the min-bound for $n_1 = 1, n_2 = 1, n_3 = 1$ (and indeed for any values for which $n_3 \leq \min(n_1, n_2)$). It is easy to see that this is an extreme point of the convex (non-bounded) polytope specified by (7).

Generally, having to deal with a dual polytope that is not non-negative causes problems and leads to complications in calculating the min-bound, though some of these potential issues disappear in this context. The feasible region is always non-empty in our case: $(\frac{1}{R}, \frac{1}{R}, \frac{1}{R}, \ldots, \frac{1}{R})$ is always in it for example, and the region is bounded above: $y_r \leq 1$ for all r. It follows by Dantzig [1998] that there is a solution of the dual problem and, indeed, it equals the solution of the primal. Thus, the min-bound is exactly the solution of the dual problem; once the extreme points of the dual polytope are found it is a simple matter to take their dot products with the sensor measurement vector $\mathbf{n}$ and obtain the maximum such number to obtain the min-bound. This means that the polytope, or rather its set of extreme points, provides the computational machine needed. We call this polytope the *fusion polytope* for the given sensor configuration and denote it by $\mathcal{C}(\mathbf{S})$. It is, obviously, only dependent on the sensor configuration. We state this key result as a theorem.

THEOREM 3.3. *Given a sensor configuration* $\mathbf{S}$*, with fusion polytope* $\mathcal{C}(\mathbf{S})$*, for any sensor measurement vector* $\mathbf{n}$*,*

$$\mathfrak{m}(\mathbf{S}, \mathbf{n}) = \max\{\mathbf{n} \cdot \mathbf{e} : \mathbf{e} \textit{ is an extreme point of } \mathcal{C}(\mathbf{S})\}. \tag{8}$$

We reiterate that, by passage to the dual linear programming problem, the calculation of min-bounds is reduced to testing certain "universal" vertices against the sensor readings. Once the extreme points of the fusion polytope are identified, a formula for the minimum that involves sensor readings as variables is immediate. Indeed some extreme points are redundant in this calculation: for instance if $\mathbf{e}$ and $\mathbf{e}'$ are extreme points and $e'_i \geq e_i$ for all i then there is no need to include $\mathbf{e}$ in the maximization in (8). We say that $\mathbf{e}$ is a *dominant* extreme point if there is no such $\mathbf{e}'$. Evidently it is only necessary to list all dominant extreme points for the purposes of (8).

Despite the existence of solutions to the dual problem, the calculation of (dominant) extreme points of the fusion polytope appears hard in general, and we currently have no solution that applies across most situations. Even in some simple cases resulting from graphical models, studied by Reed and Linhares-Sales [2003], maximization of a linear objective function is known to be NP-complete.

It turns out, somewhat surprisingly, that the issue of the fusion polytope not residing in the positive orthant is exactly that of non-genericity.

THEOREM 3.4. *Let the sensor configuration* $\mathbf{S}$ *be generic, and let* $\mathcal{C}(\mathbf{S})$ *denote the fusion polytope specified by* (5). *Then*

$$\max_{\mathbf{c}\in\mathcal{C}(\mathbf{S})} \mathbf{n}\cdot\mathbf{c} = \max_{\mathbf{c}\in\mathcal{C}(\mathbf{S})\cap P} \mathbf{n}\cdot\mathbf{c} \tag{9}$$

where P *is the positive orthant:* $P = \{\mathbf{y} : y_r \geq 0 \text{ for all } r\}$. *The converse of this statement is true. If* (9) *is true then the sensor configuration is generic.*

PROOF. Fix $\mathbf{n}$ and suppose that $\widetilde{\mathbf{y}}$ is an extreme point of $\mathcal{C}(\mathbf{S})$ for which

$$\mathbf{n}\cdot\widetilde{\mathbf{y}} = \max_{\mathbf{c}\in\mathcal{C}(\mathbf{S})} \mathbf{n}\cdot\mathbf{c}, \tag{10}$$

and assume that $\widetilde{\mathbf{y}} \notin P$. Suppose further that there is no extreme point of $\mathcal{C}(\mathbf{S})$ that is non-negative, at which the maximum is achieved. We may assume, without loss of generality that the first s coordinates $\widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2, \ldots, \widetilde{\mathbf{y}}_s$ of $\widetilde{\mathbf{y}}$ are all negative and that all subsequent coordinates of $\widetilde{\mathbf{y}}$ are non-negative. Let $\widetilde{\mathbf{y}}'$ be the vector $\widetilde{\mathbf{y}}$ with the first s coordinates replaced by 0.

Let A be the defining matrix for the linear programming problem as in (4). The generic property entails that if $\mathbf{a}$ is a row of A^T then a row vector $\mathbf{a}'$ for which the coordinatewise product satisfies $\mathbf{a}\cdot\mathbf{a}' = \mathbf{a}'$ is also a row of A^T. It follows that, for any row $\mathbf{a}$ of A^T, there is a row of A^T which has the first s rows of $\mathbf{a}$ replaced by 0. As a result, $\mathbf{a}\cdot\widetilde{\mathbf{y}}' \leq 1$ for all rows $\mathbf{a}$ of A. Thus $\widetilde{\mathbf{y}}' \in \mathcal{C}(\mathbf{S}) \cap P$. Moreover, clearly

$$\mathbf{n}\cdot\widetilde{\mathbf{y}}' \geq \mathbf{n}\cdot\widetilde{\mathbf{y}}. \tag{11}$$

The conclusion follows.

For the converse, we observe that the constraints in the dual problem are labelled and determined by atoms. In consequence $\mathcal{C}(\mathbf{S}_{\text{gen}})$ is a subset of $\mathcal{C}(S)$. However, in the case of $\mathcal{C}(S)\cap P$, only the constraints that correspond to the highest dimensional simplices (or equivalently the largest number of sensor regions in the intersection forming the atom) are relevant, since any constraint imposed in respect of such a simplex is stronger than one corresponding to a simplex contained in it. The highest dimensional constraints are common to both $\mathbf{S}$ and to $\mathbf{S}_{\text{gen}}$, so that

$$\mathcal{C}(\mathbf{S}) \cap P = \mathcal{C}(\mathbf{S}_{\text{gen}}) \cap P. \tag{12}$$

Now, for a sensor measurement vector $\mathbf{n}$,

$$\begin{aligned} \max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S})\} &\geq \max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S})\cap P\} \\ &= \max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S}_{\text{gen}})\cap P\} \\ &= \max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S}_{\text{gen}})\}. \end{aligned} \tag{13}$$

Thus if

$$\max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S})\} = \max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S})\cap P\}, \tag{14}$$

then

$$\max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S})\} = \max\{\mathbf{y}\cdot\mathbf{n} : \mathbf{y}\in\mathcal{C}(\mathbf{S}_{\text{gen}})\}. \tag{15}$$

If $\mathbf{S}$ is not generic, then there is some intersection $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_R}$ contained in a union of some different sensor regions $S_{j_1}, S_{j_2}, \ldots, S_{j_T}$.

Consider a sensor measurement vector which assigns 1 to each of the sensor regions S_{i_r} ($r = 1, 2, \ldots, R$), and 0 to every other sensor region. The min-bound for $\mathbf{S}$ is at least 2 because there can be no targets in the intersection $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_R}$. On the other hand for the corresponding sensor measurement in the generic case the min-bound is 1, since there can be just one target in the intersection $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_R}$. □

This result considerably simplifies calculations in the generic case. We call $\mathcal{C}(\mathbf{S}) \cap P$ the *positive fusion polytope*, though the word "positive" will be dropped where there is no likelihood of confusion. One simple consequence of this result is that in the generic case, if the sensor measurement increases; that is, if we have two sensor measurement vectors $\mathbf{n} = (n_1, n_2, \ldots, n_R)$ and $\mathbf{n}' = (n'_1, n'_2, \ldots, n'_R)$ where $n_i \geq n'_i$, then the minimum increases, or at least does not decrease. This may seem, at first sight obvious, but it is quickly apparent from Theorem 3.4 that it is not true in the degenerate case, and indeed this is a defining characteristic of genericity. As a simple example consider the case of Fig. 2. If the sensor measurement vector is $(1, 0, 1)$, then the minimum overall value is 2, whereas if the sensor measurement vector is $(1, 1, 1)$ the minimum overall value is 1.

In the generic case, the linear programming problem is expressible entirely in terms of the fusion complex. The translation is as follows. For each vertex of the simplicial complex (sensor region) we have the following constraint:

$$\sum_{\sigma \text{ contains vertex } v} m_\sigma = n_v, \tag{16}$$

corresponding to the constraint (3) in the "region" formulation. In addition, of course $m_\sigma \geq 0$. We need to minimize

$$\sum_\sigma m_\sigma, \tag{17}$$

subject to these constraints. The dualization results in the following problem expressed in terms of the geometry of the fusion complex.

$$\max\{\sum_r y_r : \sum_{r\in\sigma} y_r \leq 1,\ (\sigma \in \Sigma),\ y_r \geq 0\ (r = 1, 2, \ldots, R)\}. \tag{18}$$

The fusion polytope is then, with some abuse of notation,

$$\mathcal{C}(\Sigma) = \{\mathbf{y} = (y_r) : \sum_{r\in\sigma} y_r \leq 1,\ (\sigma \in \Sigma),\ y_r \geq 0\ (r = 1, 2, \ldots, R)\}. \tag{19}$$

Observe that if $\sigma \subset \sigma'$ are simplices then $\sum_{r\in\sigma} y_r \leq \sum_{r\in\sigma'} y_r$ so that the only inequalities that need be considered in defining $\mathcal{C}(\Sigma)$ are those that are maximal; that is, are not contained in any larger simplex. For instance, in the case of three regions with all possible intersections, so that the simplicial complex is the triangle and all of its subsimplices, the only inequality (other than non-negativity) that is needed is $y_1 + y_2 + y_3 \leq 1$. It follows immediately that the extreme points of Σ in this case are $(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)$ and the minimal count is $\max(n_1, n_2, n_3)$.

As another example consider the case where the simplicial complex Σ consists of the faces of a tetrahedron. This corresponds to four sensor regions such that each triple intersection is non-empty, but the quadruple intersection is empty. The constraint matrix for the dual problem (corresponding only to highest dimensional simplices) is

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

It is fairly easy to see that the fusion polytope in this case has extreme points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and so the min-bound is

$$\max(n_1, n_2, n_3, n_4, \frac{1}{3}(n_1 + n_2 + n_3 + n_4)). \tag{21}$$

4. GRAPHS

There are a number of cases where the minimum estimation problem can be posed in terms of constructs on graphs. This happens in various ways. Even when such a reformulation is possible, it does not always provide a feasible approach to computation of the min-bound; rather these formulations and the effort in the graph theory community to solve the corresponding problems suggest that the min-bound problem is, in general, difficult.

4.1. The Fractional Stable Set

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. A set of vertices $U \subset \mathcal{V}$ is called a *stable set* if no two distinct elements of U have an edge joining them; that is, if $u_1 \neq u_2$, $u_1, u_2 \in U$, then $(u_1, u_2) \notin \mathcal{E}$. The convex hull of the set of indicator functions of stable sets,

$$\mathrm{STAB}(G) = \mathrm{co}\{\mathbf{1}_U : U \text{ is a stable set of vertices}\}. \tag{22}$$

is called the *stable set polytope*. The *fractional stable set polytope* is

$$\mathrm{FRAC}(G) = \{(x_v)_{v\in\mathcal{V}} : x_v \geq 0,\ x_{v_1} + x_{v_2} \leq 1,\ (v, v_1, v_2 \in \mathcal{V}),\ (v_1, v_2) \in \mathcal{E}\}. \tag{23}$$

It is straightforward to see, and well known, that $\mathrm{STAB}(G) \subset \mathrm{FRAC}(G)$. It is also clear that the latter is exactly the fusion polytope for the situation where the R sensor regions correspond to the vertices of the graph, pairwise intersections correspond to edges, and there are no triple or higher intersections; that is, where the fusion complex is just a graph. In this context some results exist; a good reference is Wagler [2003]. Even this relatively simple case appears to produce no straightforward algorithm for computation of the extreme points. Here is an important theorem.

THEOREM 4.1 (GRÖTSCHEL, LOVÁSZ, SCHRIJVER, [GRÖTSCHEL ET AL. 1993]).

(*1*) $\mathrm{STAB}(G) = \mathrm{FRAC}(G)$ *if and only if G is bipartite with no isolated vertices;*
(*2*) *The extreme points of* $\mathrm{STAB}(G)$*, regarded as functions on the vertices of G, all have values* 0 *and* 1*;*
(*3*) *The extreme points of* $\mathrm{FRAC}(G)$ *all have values* 0*,* $1/2$*,* 1.

We illustrate these ideas with the following simple example.

Example 4.2. We consider the case when the regions $S_1, S_2, \ldots, S_R$ satisfy $S_i \cap S_{i+1} \neq \emptyset$, $S_1 \cap S_R \neq \emptyset$ and these are the only non-empty intersections. The fusion complex has R vertices connected in a cycle. The fusion polytope consists of functions on the set of vertices. A collection of extreme points of the fusion polytope consists of functions with three values: 0, 1 and $1/2$, and satisfying the following rules:

(1) every 1 is isolated, that is, two adjacent vertices cannot both be assigned a 1;
(2) no functions with three adjacent 0s appear.

Thus, between every pair of 1s there is either a single 0 or a pair of 0s. If R is even, all dominant extreme points are of this form, but if R is odd there is an additional extreme point $\mathbf{d}$ in which each vertex is assigned the value $1/2$. When R is even $\mathbf{d}$ is a convex combination of two 0-1 extreme points and so is not extreme.

While the preceding discussion shows how graph theoretic ideas apply for the case where there are no non-trivial triple intersections, there are some other situations where graph theory is applicable. A construct on graphs related to the stable and the fractional stable set polytopes is obtained as follows. We recall that a *clique* in the graph G is a set Q of vertices such that each pair of vertices in Q is connected by an edge. We define

$$\mathrm{QSTAB}(G) = \{(x_v)_{v\in\mathcal{V}} : x_v \geq 0,\ \sum_{v\in Q} x_v \leq 1,\ Q \text{ is a clique in } G\} \tag{24}$$

The QSTAB construction permits representation of the fusion polytope for other sensor configurations than just those with only pairwise intersections. For instance, consider Fig. 3. In this case, the 2-simplices are present and so the fusion complex is not a graph. However, the cliques of the 1-skeleton G are just the two 2-simplices and so $\mathrm{QSTAB}(G)$ is the fusion polytope. On the other hand if the sensor regions are as in Fig. 4 then the fusion complex is 1-dimensional and comprises the edges of a triangle.

In this case the single clique consists of all vertices of the graph and so QSTAB is not the same as the fusion polytope.

A *flag complex* is an abstract simplicial complex in which every minimal non-face has exactly two elements [Tits 1974]. In essence, any flag complex is the clique complex (that is, the set of all cliques) of its 1-skeleton. For example, if three edges of a 2-simplex are in the simplicial complex in question, then so is the 2-simplex itself. More generally, if all of the faces of a simplex are in the simplicial complex, then so is the entire simplex. The boundary of a 2-simplex does not form a flag complex, but the simplicial complex formed from all of the faces and subfaces of a 2-simplex itself does. For instance, since set $\{1, 4\}$ is the only minimal nonface in Fig. 3, the simplicial complex is a flag complex. However, $\{1, 2, 3\}$ is a minimal nonface in Fig. 4, thus it is not a flag complex. The following theorem describes when a fusion polytope is $\mathrm{QSTAB}(G)$ for some graph G.

THEOREM 4.3. *Given sensor regions $S_1, S_2, \ldots, S_R$ that form a generic sensor configuration, the fusion polytope is identified with* $\mathrm{QSTAB}(G)$ *via the inclusion of a graph G as the* 1*-skeleton of the simplicial complex of the sensor configuration if and only if it is a flag complex.*

PROOF. We assume throughout that the sensor configuration is generic. Suppose that the simplicial complex of the sensor configuration is a flag complex K. Then K is the clique complex of its 1-skeleton G by definition. $\mathrm{QSTAB}(G)$ is determined by all of the cliques and the fusion polytope is determined by the simplicial complex. Since this is a clique complex, the fusion polytope is identified with $\mathrm{QSTAB}(G)$. For the reverse direction, assume that the fusion polytope is identified with $\mathrm{QSTAB}(G)$, and that G is the 1-skeleton of the fusion complex. The fusion polytope is determined by this simplicial complex, and $\mathrm{QSTAB}(G)$ is determined by all cliques, so there is a correspondence between simplicies and cliques, which means any simplex of the simplicial complex is a clique of G. The simplicial complex is then a flag complex by definition. □

Much work has been done on $\mathrm{QSTAB}(G)$ by many authors; see, for instance, Koster and Wagler [2006] for several references. Since we are interested in the extreme points of $\mathrm{QSTAB}(G)$ in situations when $\mathrm{QSTAB}(G)$ is the fusion polytope of a sensor configuration, the following result is important. It will require the definition of a *perfect* graph. A subgraph of G is *induced* if it is the graph obtained by taking a subset of the vertices of G and all edges from G with endpoints in the subset. A graph G is *perfect* if, for every subgraph H of G, the chromatic number (the number of different colours needed to colour the vertices of the graph so that no two adjacent vertices have the same colour) of H is equal to the size of the largest clique of H. All bipartite graphs are perfect, whereas an odd cycle is not. An example of a perfect graph is shown as in Fig. 3. It is easily seen that its largest clique has size 3 and the chromatic number is 3.

THEOREM 4.4 ([CHVÁTAL 1975]). *Let G be a graph. Then* $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ *if and only if G is perfect.*

Of course, if $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ then the dominant extreme points must all take the values 0 and 1 and are fairly easy to write down; indeed they are the indicator functions of maximal stable sets. This means that odd cycles of length greater than 3 are not perfect since we have seen that they have dominant extreme points that take the value $1/2$.

5. THE TWO DIMENSIONAL CASE

5.1. Quadruple Intersections

At this point we consider planar sensing domains. As an example, consider an "inflated tiling" of the unit square $[0, 1] \times [0, 1]$; that is, a standard regular tiling by rectangles, illustrated in Fig. 5, where the tiles are slightly inflated to create overlapping regions. To be precise, the unit square is covered by Q^2 regions as in Fig. 5. Note that there are quadruple intersections but no higher, so that the fusion complex is 3-dimensional. The regions are labelled $(S_{ij})_{i,j=1}^{Q}$ in matrix style, and the following quadruple

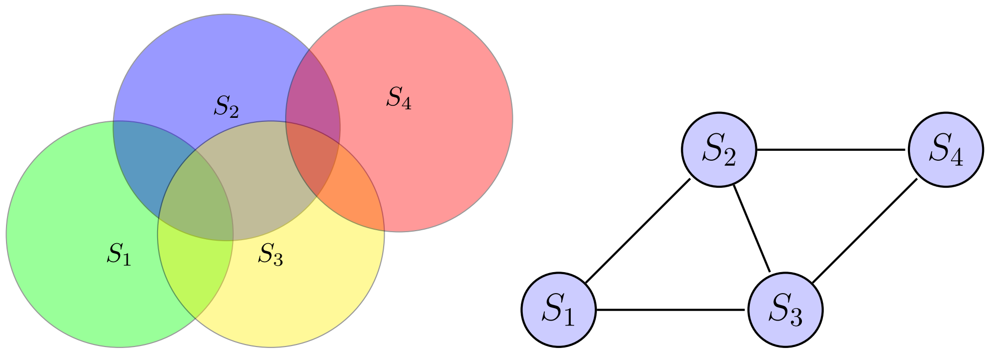


Fig. 3: Example where $QSTAB(G)$ is the fusion polytope.

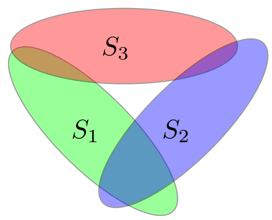


Fig. 4: Example where $QSTAB(G)$ is different from the fusion polytope.

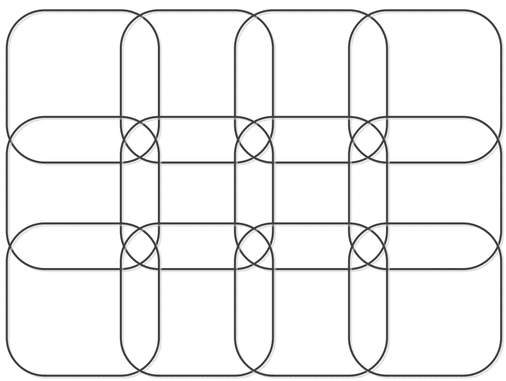


Fig. 5: Tiling by rectangles.

intersections, and only these are non-empty:

$$S_{ij} \cap S_{(i+1)j} \cap S_{i(j+1)} \cap S_{(i+1)(j+1)}. \tag{25}$$

In addition double and triple intersections forced by these quadruple ones are present. One might expect that this kind of sensor configuration would be relatively easily to handle. Unfortunately it fails to be generic, specifically there are relations such as:

$$S_{11} \cap S_{22} \subset S_{12} \cup S_{21}, \tag{26}$$

so that the potential atom $S_{11} \cap S_{22} \cap S_{12}^c \cap S_{21}^c \cap \bigcap_{i>2 \text{ or } j>2} S_{ij}^c$ is empty. It follows, by Theorem 3.4, that the fusion polytope is not in the positive orthant. Indeed the point $(-1, 1, 1, -1)$ is an extreme point of the fusion polytope for the case of just four regions $S_{11}, S_{12}, S_{21}, S_{22}$.

It turns out that this situation is typical. Indeed, any configuration of four regions in the plane that satisfies some fairly reasonable topological assumptions and that has a quadruple intersection fails to be generic, as the following theorem shows. In order to describe the result, more precision will be needed about the nature of regions and their intersections than has been required so far. At this point we introduce the topological constraints predicted when atoms were first defined in Definition 2.1.

For this purpose, a *regular* sensor region S is a bounded open set in the plane of which the boundary ∂S is a simple piecewise smooth closed curve. A *regular* sensor configuration is one comprising regular sensor regions. We shall need to redefine the notion of an *atom*. In this case, we replace each set of the form S_i^c in (2) by its interior $\overline{S_i}^c$, but we still demand that an atom is non-empty; we will use the phrase that the "atom is represented" to mean exactly that. Note that atoms are always open with this definition. A regular sensor configuration is said to be *generic* if an intersection of sensor regions is not contained in a union of different sensor regions. This is the exact analogue of generic for the non-topological case.

A regular sensor configuration $\{S_1, S_2, \ldots, S_R\}$ with $X = \cup_{r=1}^{R} S_r$ is said to be *normal* if the boundaries of any two sensor regions intersect in at most two points, and there are no points of triple intersection of boundaries of sensor regions. We also require that boundaries of atoms are simple closed curves; these are, of course made up of segments of boundaries of sensor regions. These are all fairly natural constraints. Observe that, by the Jordan-Schoenflies Theorem [Cairns 1951], these conditions imply that both closures of sensor regions and closures of atoms are homeomorphic to closed discs, and this could be taken to define the appropriate properties. For the rest of this section we assume that all sensor regions are regular and just refer to them as sensor regions.

THEOREM 5.1. *For a normal sensor configuration $\mathbf{S} = \{S_1, S_2, S_3, S_4\}$ with closure a simply-connected closed set in the plane $\mathbf{R}^2$, at least two atoms are not represented; that is, are empty. The sensor configuration is not generic.*

PROOF. To see this, construct a simplicial complex with the intersection points of boundaries of sensor regions as vertices, segments of the boundaries between the intersection points as edges, and the atoms of the sensor configuration as faces. This is a planar simplicial complex. Our assumption on intersections of boundaries of sensor regions, guarantees at most 12 vertices $(6 + 4 + 2)$ and the degree of each vertex is 4 in the graph which is the 1-skeleton of the simplicial complex. Furthermore, the number of edges is twice of the number of vertices, by the condition on the boundaries of atoms. The Euler characteristic of the closure of the union of the sensor regions, is 1, so that by Euler's formula, $v - e + f = 1$, and we have $v = f - 1$. Since $v \leq 12$, $f \leq 13$ is obtained; that is, there are at least two atoms not represented. □

If the condition of normality is relaxed to allow the boundary of one sensor region to intersect the boundary of another in 4 points, then there will be two more vertices in the corresponding graph and the number of possible faces will be equal to 15. All atoms can then be represented and connected, as the example in Fig. 6 demonstrates. Furthermore, if more pairwise intersections of boundaries of sensor regions with 4 points appear, all atoms still can still be represented, however, will be disconnected as shown in Fig. 7.

Despite this failure of genericity for the inflated rectangular tilings it is still possible to use the dual linear programming technique in this case. By Theorem 3.4, the fusion polytope is no longer in the positive orthant and the simplifications associated with that property are not available. It is still possible, however, to place a lower bound on the polytope as the following result indicates, and the region need not be rectangular for this to happen. The following notation will be useful:

$$D_\epsilon = \{\mathbf{x} \in \mathbf{R}^2 : \|\mathbf{x} - I^2\| < \epsilon\} \tag{27}$$

and $I^2 = [0, 1]^2$ is the unit square. We can use any one of a number of (convex) norms, the key features being that

$$D_\epsilon \cap \big((0, 1) + D_\epsilon\big) \cap \big((1, 0) + D_\epsilon\big) \cap \big((1, 1) + D_e\big) \neq \emptyset. \tag{28}$$

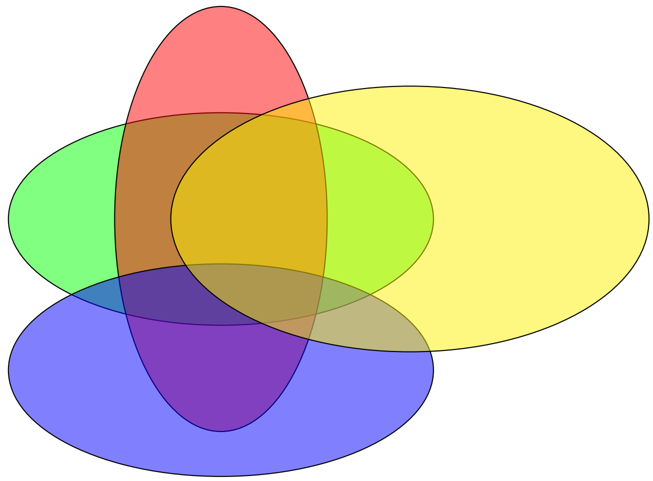


Fig. 6: Four regions in the plane with all atoms represented and connected.

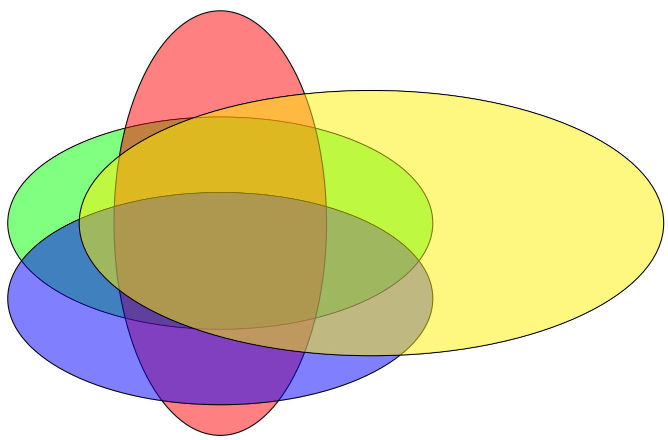


Fig. 7: Four regions in the plane with all atoms represented.

Also ϵ needs to be small enough to prohibit intersections of the form

$$D_\epsilon \cap \big((2,0) + D_\epsilon\big) \text{ or } D_\epsilon \cap \big((0,2) + D_\epsilon\big).$$

THEOREM 5.2. *Let T be a region in the plane that is a union of inflated squares based on the integer lattice. Thus*

$$T = \bigcup_{r=1}^{R} (m_r, n_r) + D_\epsilon, \tag{29}$$

and let the sensor regions be $(m_r, n_r) + D_\epsilon$. Then no point in the corresponding fusion polytope has a coordinate less than -1.

PROOF. The constraints of the fusion polytope correspond to non-empty intersections of the sensor regions, and these are of four kinds (only the translating integer pairs are listed):

— *Squares*: $\sigma = \{(n, m), (n, m + 1), (n + 1, m), (n + 1, m + 1)\}$.

— *Corners*: $\ell = \{(n,m),(n,m+1),(n+1,m)\}$ and all variants of this by rotations through 90° and reflections.
— *Segments*:$\gamma = \{(n,m),(n,m+1)\}$ and all rotations through 90°.
— *Singletons:* $\alpha = \{(n,m)\}$.

Thus, for a square we obtain the constraint,

$$x_{(n,m)} + x_{(n,m+1)} + x_{(n+1,m)} + x_{(n+1,m+1)} \leq 1, \tag{30}$$

with similar constraints for the corners and segments. Singletons give rise to the constraint $x_{(n,m)} \leq 1$.
Write A for the matrix whose rows are obtained from the constraints. The columns of A are indexed by $r = 1, \dots, R$. The fusion polytope is then $\mathcal{C} = \{\mathbf{x} \in \mathbf{R}^2 : A\mathbf{x} \leq \mathbf{1}\}$.
The extreme points of $\mathcal{C}$ are all obtained as the unique solutions of equations of the form $A_1\mathbf{x} = \mathbf{1}$ where A_1 is obtained by choosing rows from A that form a basis for $\mathbf{R}^R$. In addition, of course, $\mathbf{x} \in \mathcal{C}$, so $A_1^c\mathbf{x} \leq \mathbf{1}$ for the remaining rows A_1^c of A. We write $W(A_1)$ for the collection of squares, corners, segments and singletons corresponding to the rows of A_1.
Now suppose that the (n_r, m_r)th coordinate is less than -1 and let F be a region in $W(A_1)$ containing (n_r, m_r). Clearly, F cannot be a singleton. Neither can it be a segment or a corner since the sum over F has to be 1 and each $x_{(n,m)} \leq 1$. Suppose then that F is a square. This would imply that the sum over the members of the square other than (n_r, m_r) is at least 2. But these form a corner and so the sum has to be less than 1. This provides a contradiction and proves the result. □

Even though $\mathcal{C}$ is not bounded we can still use its extreme points to count. The following result follows quickly from standard results on convex polyhedra [Gass 2010].

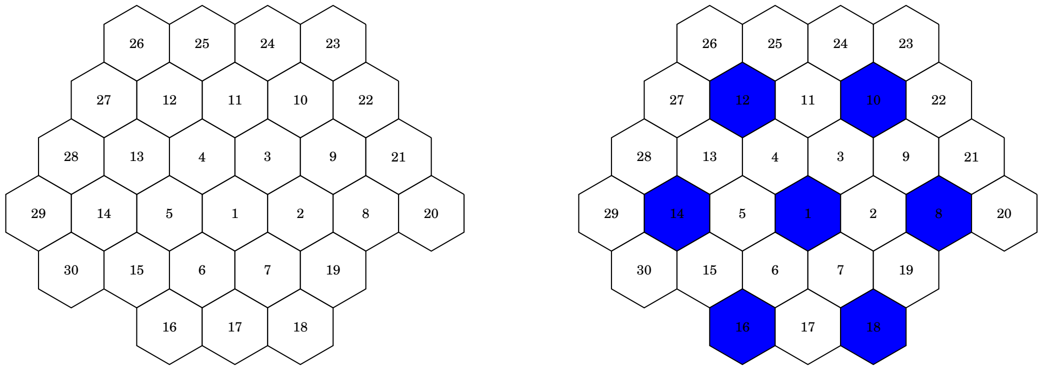
COROLLARY 5.3. *If* $\mathbf{c}$ *is a vector in* $\mathbf{R}^N$ *in the positive quadrant then*

$$\sup\{\mathbf{c} \cdot \mathbf{x} : \mathbf{x} \in \mathcal{C}\} = \max\{\mathbf{c} \cdot \mathbf{e} : \mathbf{e} \in \operatorname{Ext} \mathcal{C}\} \tag{31}$$

where $\operatorname{Ext} \mathcal{C}$ *is the set of extreme points.*

5.2. Hexagons

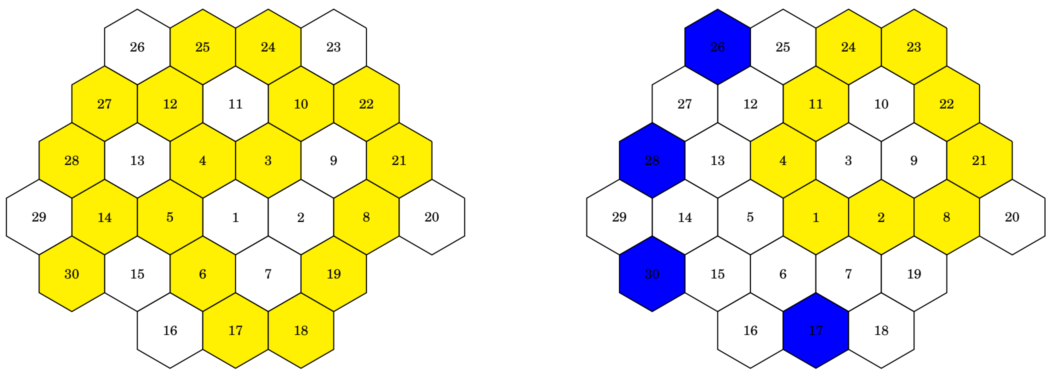
If, instead of using inflated rectangular tilings, we employ inflated hexagonal tilings, the genericity problems disappear. Let H be the regular hexagon centred at the origin in $\mathbf{R}^2$ and with vertices at $(0, \pm 1)$,$(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ and consider translates of this via the hexagonal lattice. These tesselate the plane in a honeycomb arrangement. Slight inflations of them have at most 3-fold intersections and so do not suffer degeneracy problems of rectangular tilings. In this case for a cover of some region in the plane by translates of these inflated hexagons the fusion polytope lies in the positive orthant. Even so, for large numbers of such inflated hexagons the fusion polytope can become very complicated and its number of extreme points large. We illustrate this with a few examples computed using the polymake package (`http://www.polymake.org/doku.php`). This is a topic we intend to return to in greater depth in a later paper.
Consider the tesselation of a region of the plane in Fig. 8(a). In fact, of course, we are interested in slight inflations of these hexagons so that the regions overlap.
This, incidentally, is almost as large a collection of hexagons as we are reasonably able to handle on a laptop computer running `polymake` in the space of a few days. According to polymake, the fusion polytope for the collection of 30 hexagons depicted here has nearly 5000 dominant extreme points. Many, in fact most, of these are extreme points involving only 0 (white), and 1 (blue) coefficients such as the one in Fig. 8(b). The rule for generating such (dominant) extreme points is fairly simple. No two adjacent hexagons can have a coefficient 1, and the number of 1s is maximal subject to this constraint.
More interesting are extreme points with coefficients of $1/2$ (yellow) and 0 (white), such as in Fig. 9(a).
Here again it would not be too hard to develop a description of the patterns, as we intend to do in a later paper. It is clear that no three hexagons meeting at a point can all have coefficient $1/2$. Moreover, and these appear to be related to Theorem 4.1 [Grötschel et al. 1993], every example of this kind



(a) Hexagonal tesselation of region in plane: 30 hexagons.

(b) Extreme point with $0, 1$ coefficients.

Fig. 8: Tesselation and extreme point involving only $0, 1$ coefficients.



(a) Extreme point with $0, \frac{1}{2}$ coefficients.

(b) Mixture of $0, 1$ and $0, \frac{1}{2}$ extreme point.

Fig. 9: Extreme points involving $0, \frac{1}{2}, 1$ coefficients.

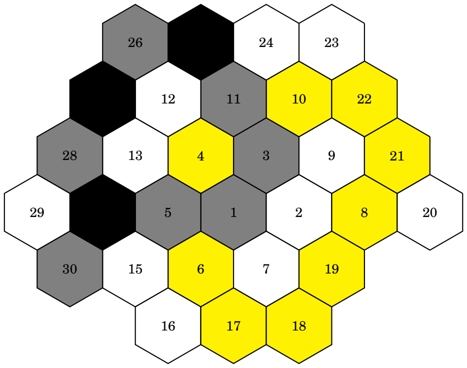
involves a loop with an odd number of hexagons. There are, however, other possibilities for extreme points. One rather obvious one is a mixture of the preceding two types as is illustrated in Fig. 9(b).

More exotic extreme points also arise. To illustrate, consider Fig. 10(a) (where grey represents $\frac{1}{4}$ and black $\frac{3}{4}$, and yellow, as before, represents $\frac{1}{2}$), and Fig. 10(b) (where green represents $\frac{1}{3}$ and red $\frac{2}{3}$). Coefficients other than $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$ have also been observed.

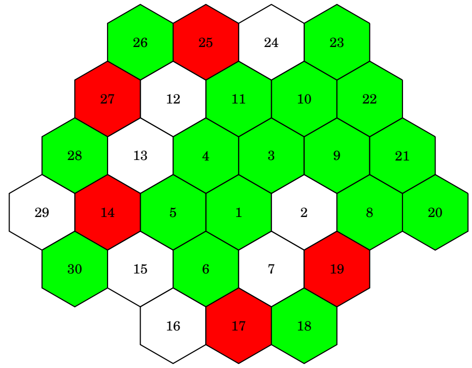
6. BOUNDS AND SPECIAL TYPES OF REGIONS

We consider here only (irredundant, covering,) generic sensor configurations. These are entirely represented by the fusion complex and so are discussed in those terms. Accordingly, we fix a fusion complex Σ for sensor configuration $\mathbf{S} = (S_1, S_2, \ldots, S_R)$. The k-*skeleton* of Σ is defined to be the subcomplex of Σ consisting of those subsets of size $\leq k+1$ (or in other words simplices of dimension $\leq k$ and denoted by Σ^k. We write Σ_k for the simplicial complex obtained from Σ by inserting a new simplex whenever Σ contains all of the k-faces of that simplex. In other words, if every subset of $\sigma = (i_1, i_2, \ldots, i_r)$ $(1 \leq i_n \leq R)$ of size $k+1$ belongs to Σ then $\sigma \in \Sigma$.

The following result is a straightforward observation. Recall that genericity permits us to consider the fusion polytope as a subset of the positive orthant.



(a) Extreme point with coefficients $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$.



(b) Extreme point with coefficients $0, \frac{1}{3}, \frac{2}{3}$.

Fig. 10: Exotic extreme points.

THEOREM 6.1.

$$\mathcal{C}(\Sigma_1) \subset \mathcal{C}(\Sigma_k) \subset \mathcal{C}(\Sigma) \subset \mathcal{C}(\Sigma^k) \subset \mathcal{C}(\Sigma^1), \tag{32}$$

and so for any measurement vector $\mathbf{n}$*,*

$$\max \mathbf{n} = \mathfrak{m}(\Sigma_1, \mathbf{n}) \leq \mathfrak{m}(\Sigma_k, \mathbf{n}) \leq \mathfrak{m}(\Sigma, \mathbf{n}) \leq \mathfrak{m}(\Sigma^k, \mathbf{n}) \leq \mathfrak{m}(\Sigma^1, \mathbf{n}) = \mathbf{1}.\mathbf{n}. \tag{33}$$

Of course Σ^2 is a graph and so the techniques discussed in Section 4 can be applied to the calculation of $\mathfrak{m}(\Sigma^2, \mathbf{n})$.

Some kinds of restrictions on shapes make calculating and bounding the minimal count easier. When the regions are convex subsets of $\mathbf{R}^n$, Helly's Theorem can be applied. We state this as:

THEOREM 6.2 ([HELLY 1923]). *If $C_1, C_2, \ldots, C_R$ are convex subsets of $\mathbf{R}^n$ with the property that $C_{i_1} \cap C_{i_2} \cap \cdots \cap C_{i_{n+1}} \neq \emptyset$ for all choices of i_k, then $C_1 \cap C_2 \cap \cdots \cap C_R \neq \emptyset$.*

In terms of the simplicial complex associated with the sensor regions this yields the following property.

THEOREM 6.3. *Let $\mathbf{S} = (S_1, S_2, \ldots, S_R)$ be a generic sensor configuration for which the sensor regions are convex subsets of $\mathbf{R}^n$. Let Σ be the corresponding simplicial complex. Then Σ contains all of the N-dimensional faces of a simplex it must also contain the simplex. In other words $\Sigma_N = \Sigma$.*

In particular, if the sensor regions are in the plane $\mathbf{R}^2$ then $\Sigma_2 = \Sigma$.

Another situation in which it is possible to make further progress concerns a generic collection of sensor regions with the property that every sensor region contains a "uniformly maximal dimensional intersection"; in other words, there is some M such that for every r,

$$S_r \supset S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{M+1}} \neq \emptyset,$$

but there are no non-trivial non-empty $M + 2$ intersections. This amounts to a "manifold" assumption on the fusion complex; namely, that every simplex is a subset of one of maximal dimension M. In this case, in the dual formulation of the linear programming problem, the only inequalities that need to be considered are the ones involving the maximal dimensional simplices; that is, $\mathcal{C}(\Sigma) = \mathcal{C}(\Sigma_M)$. The linear programming problem is significantly simplified because of the reduced number of inequalities. To illustrate this we present the following example.

Let $\sigma_a = \{a, b, c, d\}$, $\sigma_r = \{r, s, t, u\}$, and $\sigma_w = \{w, x, y, z\}$ be three generic tetrahedra given in terms of their vertices in $\mathbf{R}^3$.

Example 6.4.

(1) First consider the fusion complex comprising σ_a and σ_r with $a = r$. In this case all extreme points have coordinates equal to either 0 or 1. Only one vertex of each tetrahedron can be assigned a 1. This is the only restriction, resulting in 10 (dominant) extreme points.
(2) Now consider the fusion complex where the three tetrahedra are attached in a string: $a = r$ and $u = w$. As the same as the previous case, all extreme points have coordinates equal to either 0 or 1.
(3) Next, suppose that the three tetrahedra are connected in a cycle: $a = r$ and $u = w$, $x = b$. Then, in addition to the obvious $0, 1$ extreme points, there is one $0, \frac{1}{2}$ extreme point with values $\frac{1}{2}$ at a, u, and x.
(4) Now suppose that the tetrahedra are connected along edges, say, $(a, b) = (r, s)$ and $(t, u) = (w, x)$. then again all extreme points are $0, 1$-valued.
(5) Finally consider the case $(a, b) = (r, s)$, $(t, u) = (w, x)$, $(y, z) = (c, d)$. In this case there also $0, \frac{1}{2}$-valued extreme points with supports triangles with one edge in each tetrahedron.

Surprisingly, even if more tetrahedra are glued together in this way, the coordinates of all extreme points still only can be 0 or 1, although the number of them increases quickly. We will discuss these conclusions in much more detail in a later paper.

7. PROCEDURE AND EXAMPLE

In summary, given a generic sensor configuration $\mathbf{S} = \{S_1, S_2, \ldots, S_R\}$, and sensor measurements $\mathbf{n} = (n_1, n_2, \ldots, n_R)$, the procedure for calculation of the minimum number of targets is as follows.

— List the constraint inequalities in (5) via the sensor configuration $\mathbf{S}$, using only the sensor intersection structure, and obtain the sensor fusion polytope $\mathcal{C}(\mathbf{S})$;
— Calculate all dominant extreme points $\mathbf{e}$ of the fusion polytope $\mathcal{C}(\mathbf{S})$, store them in a database and denote this set by set P;
— For sensor measurements $\mathbf{n}$, the minimum number of targets is $\max_{\mathbf{e} \in P} \mathbf{n} \cdot \mathbf{e}$;

From the above steps and the preceding sections, for a specific sensor configuration $\mathbf{S}$, the dominant extreme points remain invariant to changes in sensor measurements. Since we can regard extreme points of a sensor configuration as a prior and calculate them offline, the online computation cost depends only on the number of extreme points and therefore potentially significantly reduces computations.

To illustrate this, we assume the sensor configuration shown in Fig. 11. In this sensor configuration, there are 19 sensors with up to triple intersections. This can easily be seen to be generic. Using an extreme point enumeration algorithm, we can show that the fusion polytope of this sensor configuration has 163 dominant extreme points, not listed here. For any sensor measurements $\mathbf{n} = (n_1, n_2, \ldots, n_{19})$ at time t, the minimum number of targets is the maximum value of the dot products of sensor measurements $\mathbf{n}$ and each extreme point. The statistical results of $10,000$ Monte Carlo simulations for this example are listed in Table I . For each setting of sensor's detection radius r and distance d between two neighbouring sensors, we let the number of targets be different values and the targets be randomly uniformly distributed in the sensing domain. We then calculate the relative ratio of the derived minimum number of targets to the true number of targets, i.e., $\frac{N_T - \bar{N}}{N_T}$, where N_T is the true number of targets and $\bar{N}$ is the minimum number of targets estimated by the procedure listed above.

Table I: Estimation of minimum number of targets in sensor configuration in Fig. 11.

number of targets	50	100	200	300	400	500	600	700	800	900	1000
$r = 2, d = 2.5$	0.288	0.309	0.319	0.323	0.326	0.327	0.328	0.329	0.330	0.331	0.331
$r = 2, d = 2.7$	0.346	0.370	0.383	0.387	0.389	0.391	0.392	0.393	0.394	0.395	0.396
$r = 2, d = 3.0$	0.419	0.447	0.461	0.466	0.469	0.471	0.472	0.472	0.473	0.474	0.474

From Table I, it is seen that, for a fixed sensor radius and number of targets, the relative ratio of the derived minimum number of targets to the true number of targets will statistically increase with

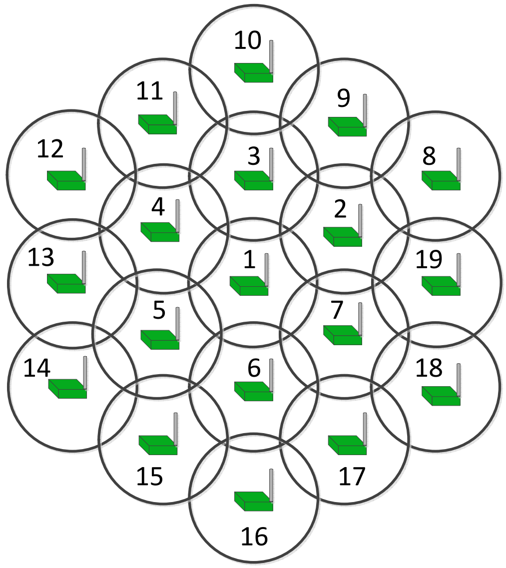


Fig. 11: Sensor configuration with 19 sensors.

the increasing of the distance between two neighbouring sensors. This is because more targets are in the non-overlapping areas of the sensors so that the reported number of targets for each sensor are comparably small. For a fixed sensor radius and the distance between two neighbouring sensors, the relative ratio of the derived minimum number of targets to the true number of targets will gradually increase up to a limit value when the number of targets increases. The reason for that is, when the number of targets increases, the average number of targets in the region of sensors of which reports are used to derive the minimum number of targets will statistically decreasingly tend to the average number of targets in the entire sensing domain.

8. CONCLUSION

This paper introduces theoretical methods for obtaining the limits on fusion of simple measurements from multiple sensors. The problem is formulated as one in linear programming and dualized to obtain an object, called the fusion polytope, which is the key device for calculation of the minimum fused value compatible with the data. The fusion polytope is computed in some simple cases. It is shown that when the sensor configuration satisfies a simple property, defined in the paper as *genericity*, this fusion polytope can be assumed to be in the positive orthant.

It is also shown that, under some mild hypotheses, genericity is not satisfied when there are four overlapping sensor regions in the plane. Consideration is given to the case of overlapping regions in a hexagonal tiling in the plane. This is a generic situation, but simulations have shown that it still leads to very complex extreme point structures.

We believe that the ideas described here are capable of being extended to more complex situations where data fusion is required.

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REFERENCES

Shuchin Aeron, Manqi Zhao, and Venkatesh Saligrama. 2006. Fundamental tradeoffs between sparsity, sensing diversity and sensing capacity. In *Proceedings of the 40th Asilomar Conference on Signals, Systems and Computers*. IEEE Computer Society, Piscataway, NJ, USA, 295–299. DOI:http://dx.doi.org/10.1109/ACSSC.2006.356635

Paul Alexandroff and Heinz Hopf. 1935. *Topologie*. Springer, Berlin.

Yuliy M. Baryshnikov, Edward G. Coffman, Kyung J. Kwak, and Bill Moran. 2008. Stochastic counting in sensor networks, or: Noise is good. In *Distributed Computing in Sensor Systems - 4th IEEE International Conference, DCOSS 2008, Proceedings*. Springer Verlag, Heidelberg, Germany, 32–45. DOI:http://dx.doi.org/10.1007/978-3-540-69170-9_3

Yassine Benabbas, Nacim Ihaddadene, Tarek Yahiaoui, Thierry Urruty, and Chabane Djeraba. 2010. Spatio-temporal optical flow analysis for people counting. In *Proceedings of 2010 Seventh IEEE International Conference on Advanced Video and Signal Based Surveillance (AVSS)*. IEEE Computer Society, Piscataway, NJ, USA, 212–217. DOI:http://dx.doi.org/10.1109/AVSS.2010.29

Stewart S. Cairns. 1951. An elementary proof of the Jordan-Schoenflies theorem. *Proc. Amer. Math. Soc.* 2, 6 (1951), 860–867. DOI:http://dx.doi.org/10.2307/2031698

Holun Cheng and Tony Tan. 2004. Subdividing alpha complex. In *Proceedings of FSTTCS 2004: Foundations of Software Technology and Theoretical Computer Science*, Vol. 3328. Springer Berlin Heidelberg, Heidelberg, Germany, 186–197. DOI:http://dx.doi.org/10.1007/b104325

Václav Chvátal. 1975. On certain polytopes associated with graphs. *Journal of Combinatorial Theory, Series B* 18, 2 (1975), 138–154. DOI:http://dx.doi.org/10.1016/0095-8956(75)90041-6

Michael L. Connolly. 1985. Computation of molecular volume. *Journal of the American Chemical Society* 107, 5 (March 1985), 1118–1124. DOI:http://dx.doi.org/10.1021/ja00291a006

George B. Dantzig. 1998. *Linear Programming and Extensions*. Princeton University Press, New Jersey.

Qing Fang, Feng Zhao, and Leonidas Guibas. 2002. *Counting targets: Building and managing aggregates in wireless sensor networks*. Technical Report P2002-10298. Palo Alto Research Center (PARC), Palo Alto, CA, USA.

Sorabh Gandhi, Rajesh Kumar, and Subhash Suri. 2008. Target counting under minimal sensing: Complexity and approximations. In *Algorithmic Aspects of Wireless Sensor Networks: Fourth International Workshop, ALGOSENSORS 2008, Revised Selected Papers*. Springer-Verlag, Heidelberg, Germany, 30–42. DOI:http://dx.doi.org/10.1007/978-3-540-92862-1_4

Saul I. Gass. 2010. *Linear Programming: Methods and Applications* (fifth ed.). Dover Publications, New York, NY.

Robert Ghrist and Yuliy Baryshnikov. 2008. Target enumeration via integration over planar sensor networks. In *Proceedings of Robotics: Science and Systems IV*. The MIT Press, Cambridge, Massachusetts, USA, 1–8.

Martin Grötschel and László Lovász. 1995. Combinatorial Optimization. In *Handbook of Combinatorics*, Ronald L. Graham, Martin Grötschel, and László Lovász (Eds.). Elsevier, North-Holland, Netherlands, Chapter 28, 1541–1598.

Martin Grötschel, Lászlo Lovász, and Alexander Schrijver. 1993. Geometric Algorithms and Combinatorial Optimization. In *Algorithms and Combinatorics*. Springer-Verlag, Berlin, Germany.

Shuo Guo, Tian He, Mohamed F. Mokbel, John A. Stankovic, and Tarek F. Abdelzaher. 2008. On accurate and efficient statistical counting in sensor-based surveillance systems. In *Proceedings of 5th IEEE International Conference on Mobile Ad-Hoc and Sensor Systems, MASS 2008*. IEEE, Piscataway, NJ, USA, 24–35. DOI:http://dx.doi.org/10.1109/MAHSS.2008.4660038

Shuo Guo, Tian He, Mohamed F Mokbel, John A Stankovic, and Tarek F Abdelzaher. 2010. On accurate and efficient statistical counting in sensor-based surveillance systems. *Pervasive and Mobile Computing* 6, 1 (2010), 74–92. DOI:http://dx.doi.org/10.1016/j.pmcj.2009.07.013

Eduard Helly. 1923. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 32 (1923), 175–176.

Xuegang Huang and Hua Lu. 2007. Snapshot density queries on location sensors. In *Proceedings of the 6th ACM international workshop on Data engineering for wireless and mobile access*. ACM, New York, NY, USA, 75–78. DOI:http://dx.doi.org/10.1145/1254850.1254864

Arie M.C.A. Koster and Annegret K. Wagler. 2006. *The extreme points of QSTAB(G) and its implications*. Technical Report 06-30. Zuse Institute Berlin, Berlin, Germany.

Loukas Lazos and Radha Poovendran. 2006. Stochastic coverage in heterogeneous sensor networks. *ACM Transactions on Sensor Networks* 2, 3 (August 2006), 325–358. DOI:http://dx.doi.org/10.1145/1167935.1167937

Loukas Lazos, Radha Poovendran, and James A. Ritcey. 2007. Probabilistic detection of mobile targets in heterogeneous sensor networks. In *Proceedings of the 6th international conference on information processing in sensor networks*. ACM, New York, NY, USA, 519–528. DOI:http://dx.doi.org/10.1145/1236360.1236426

Jingwen Li, Lei Huang, and Changping Liu. 2012. People counting across multiple cameras for intelligent video surveillance. In *Proceedings of 2012 IEEE Ninth International Conference on Advanced Video and Signal-Based Surveillance (AVSS)*. IEEE Computer Society, Washington, DC, USA, 178–183. DOI:http://dx.doi.org/10.1109/AVSS.2012.54

Jie Liang. 2007. Computation of protein geometry and its applications: Packing and function prediction. In *Computational Methods for Protein Structure Prediction and Modeling*, Ying Xu, Dong Xu, and Jie Liang (Eds.). Springer New York, New York, USA, 181–206. DOI:http://dx.doi.org/10.1007/978-0-387-68372-0_6

Xingwu Liu, Juhua Pu, and Jianzhong Pan. 2008. A classification of degenerate loop agreement. In *Proceedings of Fifth IFIP International Conference on Theoretical Computer Science-TCS 2008*, Vol. 273. Springer US, New York, NY, USA, 203–213. DOI:http://dx.doi.org/10.1007/978-0-387-09680-3

Chen Change Loy, Ke Chen, Shaogang Gong, and Tao Xiang. 2013. Crowd counting and profiling: Methodology and evaluation. In *Modeling, Simulation and Visual Analysis of Crowds*. Springer New York, New York, NY, USA, 347–382. DOI:http://dx.doi.org/10.1007/978-1-4614-8483-7_14

Jiri Matousek and Bernd Gärtner. 2007. *Understanding and using linear programming*. Springer-Verlag, Heidelberg.

Anurag Mittal and Larry S. Davis. 2004. Visibility analysis and sensor planning in dynamic environments. In *Proceedings of 8th European Conference on Computer Vision (ECCV 2004)*. Springer-Verlag, Berlin, Germany, 175–189.

Andrew Newell and Kemal Akkaya. 2009. Self-actuation of camera sensors for redundant data elimination in wireless multimedia sensor networks. In *Proceedings of the 2009 IEEE international conference on Communications*. IEEE, Piscataway, NJ, USA, 133–137. DOI:http://dx.doi.org/10.1109/ICC.2009.5199448

Lakshman Prasad, Sitharama S. Iyengar, Rangasami L. Kashyap, and Rabinder N. Madan. 1991. Functional characterization of fault tolerant integration in distributed sensor networks. *IEEE Transactions on Systems, Man, and Cybernetics* 21, 5 (Sep/Oct 1991), 1082–1087. DOI:http://dx.doi.org/10.1109/21.120060

Bruce A. Reed and Claudia L. Linhares-Sales. 2003. *Recent Advances in Algorithms and Combinatorics*. Springer, New York.

Alexander Schrijver. 1995. Polyhedral Combinatorics. In *Handbook of Combinatorics*, Ronald L. Graham, Martin Grötschel, and László Lovász (Eds.). Elsevier, Berlin, Germany, Chapter 30, 1649–1704.

Jacques Tits. 1974. *Buildings of spherical type and finite BN-pairs*. Vol. 386. Springer-Verlag, Berlin, Germany.

Annegret K. Wagler. 2003. Graph theoretical problems and related polytopes: Stable sets and perfect graphs. (2003). http://www.oil.di.univaq.it/ricerca/eventi/seminarioWagler/BlockSeminar.pdf

Dengyuan Wu, Dechang Chen, Kai Xing, and Xiuzhen Cheng. 2012. A statistical approach for target counting in sensor-based surveillance systems. In *INFOCOM, 2012 Proceedings IEEE*. IEEE, Piscataway, NJ, USA, 226–234. DOI:http://dx.doi.org/10.1109/INFCOM.2012.6195613

Dengyuan Wu, Bowu Zhang, Hongjuan Li, and Xiuzhen Cheng. 2014. Target counting in wireless sensor networks. In *The Art of Wireless Sensor Networks*. Springer Berlin Heidelberg, Heidelberg, Germany, 235–269. DOI:http://dx.doi.org/10.1007/978-3-642-40066-7_7

Qishi Wu, Nageswara S. V. Rao, Xiaojiang Du, Sundaraja S. Iyengar, and Vijay K. Vaishnavi. 2007. On efficient deployment of sensors on planar grid. *Computer Communications* 30, 14-15 (October 2007), 2721–2734. DOI:http://dx.doi.org/10.1016/j.comcom.2007.05.012

Bowu Zhang, Xiuzhen Cheng, Nan Zhang, Yong Cui, Yingshu Li, and Qilian Liang. 2011. Sparse target counting and localization in sensor networks based on compressive sensing. In *INFOCOM, 2011 Proceedings IEEE*. IEEE, Piscataway, NJ, USA, 2255–2263. DOI:http://dx.doi.org/10.1109/INFCOM.2011.5935041

Chengming Zou and Edwin R. Hancock. 2010. Face recognition using simplicial complexes. In *Advances in Pattern Recognition - Second Mexican Conference on Pattern Recognition, MCPR 2010, Proceedings*. Springer, Heidelberg, Germany, 22–29. DOI:http://dx.doi.org/10.1007/978-3-642-15992-3_3