An Exact Bayesian Detector for Multistatic Passive Radar

Stephen D. Howard*, Songsri Sirianunpiboon* and Douglas Cochran†

*Defence Science and Technology Group, PO Box 1500, Edinburgh 5111, Australia
†School of Mathematical and Statistical Sciences, Arizona State University, Tempe AZ 85287-5706 USA

***Abstract*—An exact Bayesian likelihood ratio is derived for detecting the presence of a rank-2 signal in $M > 2$ channels of noisy receiver data under the assumption that the signal is known to be present on $K = 2$ of the channels (reference channels). The objective of the test is thus to ascertain whether the signal is also present on the other channels (surveillance channels). The performance of the Bayesian detector is compared to that of the generalized likelihood ratio test (GLRT). In this scenario, the Bayesian detector is found to be quite significantly better than the GLRT.**

***Index Terms*—Passive radar; Multi-channel detection; Bayesian detector; GLRT**

I. Introduction

Passive radar has been a technology of increasing interest over the past few years. Its potential features, including covertness of operation and simplicity of system design, are well documented. The absence of a need for dedicated spectral resources has also been seen as an attractive feature in view of growing attention to efficient use and management of the radio-frequency (RF) spectrum [1]. Despite favorable properties of modern digitally modulated broadcast waveforms for use in passive radar, in comparison to their predecessors, many signal processing challenges are entailed in achieving desirable levels of performance from this technology.

The focus of this paper is detection of a scattered signal in multiple surveillance channels using $K = 2$ linearly independent reference signals, each corresponding to a noisy direct-path channel from an opportunistic transmitter. The mathematical model used for this scenario, described in Section II, was introduced by the authors in [2] and has subsequently appeared with some variations in multiple papers on detection in passive radar. The generalized likelihood ratio test (GLRT) for this model was derived in [2], where an approximate Bayesian test was also formulated. The Bayesian formulation entails calculation of integrals over complex Grassmannian manifolds, which was undertaken by Laplace approximation. Subsequently, the authors have developed exact solutions for such integrals [3], and the incorporation of these results to obtain the exact form of the Bayesian detectors is the main contribution of this paper. Simulation results are provided for two scenarios. The first is the conventional bistatic case with $M = 2$ antennas, one for the direct path and one for surveillance. The second case involves $M = 3$ receive antennas, $K = 2$ of which are configured to receive direct-path signals from distinct transmitters and $M - K = 1$ of which is a surveillance antenna. In this setting, the Bayesian detector is seen to offer very substantially improved performance over both the GLRT and the multi-signal generalization of cross correlation.

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II. Model and Problem Formulation

Although our main aim is to consider $K = 2$ reference channels, in this, and part of the next section, we will leave K unspecified. The passive radar configuration to be modeled consists of $M \geq 2$ directional receive antennas. Of these, a fixed number $K < M$ are used to receive direct-path reference signals from K transmitters of opportunity; i.e., one antenna is devoted to each of the transmitters. These antennas are configured to minimize mutual interference between the received reference signals, which are assumed to be linearly independent. The other $M - K$ antennas are used as surveillance receivers. The detection problem to be addressed is, on the basis of data from all M receive channels, to distinguish between hypotheses:

- H_0: The K reference channels contain a rank-K signal plus white Gaussian noise while the $M - K$ surveillance channels contain only white Gaussian noise.
- H_1: There is a rank-K signal across all M channels plus white Gaussian noise on each channel.

Under both hypotheses, the noise is assumed to have mean zero and to be independent across the channels.

As argued in [2], in the absence of further knowledge regarding the structures and amplitudes of the transmitted signals, and assuming appropriate direct-path interference mitigation (e.g., [4]) has been applied to the surveillance channel data, the hypothesis test may be formulated as follows. Denote by $X \in \mathbb{C}^{M \times N}$ the matrix whose first K rows are data vectors from the K direct-path receivers and whose remaining $M - K$ rows contain the data from the surveillance receivers; i.e.,

$$X = \begin{pmatrix} Y \\ Z \end{pmatrix}$$

where Y is the $K \times N$ matrix of reference channel data and Z is the $(M-K) \times N$ matrix of surveillance channel data. The data vectors are assumed to have been appropriately sampled and compensated for delay and Doppler to represent the target position and velocity under consideration in the test. With this notation, the hypotheses are

$$\begin{aligned} H_1 \ : \ Y &= A_\mathrm{r} S + \nu_\mathrm{r} \\ Z &= A_\mathrm{s} S + \nu_\mathrm{s} \end{aligned}$$

and

$$\begin{aligned} H_0 \ : \ Y &= A_\mathrm{r} S + \nu_\mathrm{r} \\ Z &= \nu_s. \end{aligned}$$

In these expressions, $S \in \mathbb{C}^{K \times N}$ is a matrix with orthonormal rows that span the K-dimensional subspace of $\mathbb{C}^N$ defined by the K reference signals and $A_\mathrm{r} \in \mathbb{C}^{K \times K}$ is an unknown matrix representing the channel gains between the transmitters and reference-channel receivers. Similarly $A_\mathrm{s} \in \mathbb{C}^{(M-K)\times K}$ is an unknown matrix representing the indirect-path channel gains between the transmitters and surveillance-channel receivers under H_1. Denoting hermitian transpose by $\dagger$, note that $SS^\dagger = \mathbb{I}_K$, the $K \times K$ identity matrix. Also, $X^\dagger X = Y^\dagger Y + Z^\dagger Z$. Under both hypotheses, the noise vector

$$\nu = \begin{pmatrix} \nu_\mathrm{r} \\ \nu_\mathrm{s} \end{pmatrix}$$

is white complex Gaussian with mean zero and variance σ^2.

III. Multistatic Bayesian Detector

With the notation established in Section II, the joint probability density function (pdf) of the data Y and Z under H_0, conditioned on the parameters A_r, S, and σ^2 is

$$\begin{aligned} &p(X|H_0; A_\mathrm{r}, S, \sigma^2) = p(Y, Z|H_0; A_\mathrm{r}, S, \sigma^2) \\ &\quad = (\pi\sigma^2)^{-MN} e^{-\frac{1}{\sigma^2}\{\mathrm{Tr}(Y - A_\mathrm{r}S)(Y-A_\mathrm{r}S)^\dagger + \mathrm{Tr}(ZZ^\dagger)\}}. \end{aligned}$$

Under H_1, the joint pdf of Y and Z conditioned on the parameters A_s, A_r, S, and σ^2 is

$$\begin{aligned} &p(X|H_1; A_\mathrm{s}, A_\mathrm{r}, S, \sigma^2) = p(Y, Z|H_1; A_\mathrm{s}, A_\mathrm{r}, S, \sigma^2) \\ &= (\pi\sigma^2)^{-MN} e^{-\frac{1}{\sigma^2}\{\mathrm{Tr}(Y-A_\mathrm{r}S)(Y-A_\mathrm{r}S)^\dagger + \mathrm{Tr}(Z - A_\mathrm{s}S)(Z-A_\mathrm{s})^\dagger\}}. \end{aligned}$$

Also in [2], a Bayesian test was formulated for the hypothesis test H_1 against H_0. Specifically,

$$\begin{aligned} &\frac{p(X|H_1)}{p(X|H_0)} \\ &\quad = \frac{\int p(X|A_\mathrm{s}, A_\mathrm{r}, S, \sigma^2) p(A_\mathrm{s}, A_\mathrm{r}|\sigma^2) p(S)\, dA_\mathrm{r}\, dA_\mathrm{s}\, dS}{\int p(X|A_\mathrm{r}, S, \sigma^2) p(A_\mathrm{r}|\sigma^2) p(S)\, dA_\mathrm{r}\, dS} \end{aligned}$$

where the integrals are over the spaces on which the parameters are defined. Following [5], the prior probability densities for A_r and A_s were chosen to be independent conditioned on σ^2 with

$$p(A_\mathrm{r}|\sigma^2) = (\pi\beta_\mathrm{r}^2\sigma^2)^{-K^2} e^{-\frac{1}{\beta_\mathrm{r}^2\sigma^2}\mathrm{Tr}(A_\mathrm{r}A_\mathrm{r}^\dagger)}$$

and

$$p(A_\mathrm{s}|\sigma^2) = (\pi\beta_\mathrm{s}^2\sigma^2)^{-K(M-K)} e^{-\frac{1}{\beta_\mathrm{s}^2\sigma^2}\mathrm{Tr}(A_\mathrm{s}A_\mathrm{s}^\dagger)}$$

where β_s and β_r are dimensionless parameters that can be chosen.

With σ^2 known, the marginalized likelihood ratio takes the form

$$\begin{aligned} &\frac{p(X|H_1)}{p(X|H_0)} \\ &= \frac{\int_{G_{K,N}} \exp \frac{1}{\sigma^2}\mathrm{Tr}(P(\alpha_\mathrm{r} Y^\dagger Y + \alpha_\mathrm{s} Z^\dagger Z))\, d\mu(P)}{(1+\beta_\mathrm{s}^2)^{K(M-K)} \int_{G_{K,N}} \exp \frac{\alpha_\mathrm{r}}{\sigma^2}\mathrm{Tr}(PY^\dagger Y)\, d\mu(P)} \end{aligned} \tag{1}$$

where $\alpha_\mathrm{s} = \beta_\mathrm{s}^2/(1+\beta_\mathrm{s}^2)$, $\alpha_\mathrm{r} = \beta_\mathrm{r}^2/(1+\beta_\mathrm{r}^2)$, and $P = S^\dagger S$. Note that P is a rank-K projection matrix and the integral is over $G_{K,N}$, the complex Grassmannian manifold of K-dimensional subspaces in $\mathbb{C}^N$ of which P is an element. The integral is with respect to the invariant probability measure $d\mu(P)$ on $G_{K,N}$, as discussed in [3], [5]. We generally take $\beta_\mathrm{r} \to \infty$ and consequently $\alpha_\mathrm{r} \to 1$, which corresponds to a completely non-infomative prior for A_r.

IV. Bistatic Case

We begin with a brief discussion of the simple bistatic case in which $M = 2$ and $K = 1$. In this case the problem reduces to

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{s} + \boldsymbol{\nu}_\mathrm{r} \\ \boldsymbol{z} &= A\boldsymbol{s} + \boldsymbol{\nu}_\mathrm{s} \end{aligned}$$

where we test $H_1 : A > 0$ against $H_0 : A = 0$. If we have no prior knowledge of the signal $\boldsymbol{s} \in \mathbb{C}^N$ or the amplitude $A \in \mathbb{C}$, then it is interesting to note that the detection problem is invariant under the transformations

$$\boldsymbol{y} \to U\boldsymbol{y}, \quad \boldsymbol{z} \to U\boldsymbol{z} \quad \text{and} \quad \boldsymbol{s} \to U\boldsymbol{s}$$

and

$$\boldsymbol{z} \to e^{i\phi}\boldsymbol{z} \quad \text{and} \quad A \to e^{i\phi}A$$

where U is any unitary $N \times N$ matrix and $\phi \in \mathbb{R}$. Therefore, any detector should be a function of the maximal invariants for the action of the above group of transformations on the space in which that data is realized. The maximal invariants in this case are $\|\boldsymbol{y}\|^2$, $|\boldsymbol{z}^\dagger\boldsymbol{y}|^2$ and $\|\boldsymbol{z}\|^2$. The GLR and Bayesian detection statistics are functions $\|\boldsymbol{y}\|^2$ and the eigenvalues of the scaled Gram matrix

$$G = \begin{pmatrix} \|\boldsymbol{y}\|^2 & \sqrt{\alpha_\mathrm{s}}\boldsymbol{y}^\dagger\boldsymbol{z} \\ \sqrt{\alpha_\mathrm{s}}\boldsymbol{z}^\dagger\boldsymbol{y} & \alpha_\mathrm{s}\|\boldsymbol{z}\|^2 \end{pmatrix}$$

which are

$$\lambda_{1,2}^{(\alpha_\mathrm{s})} = \frac{\|\boldsymbol{y}\|^2 + \alpha_\mathrm{s}\|\boldsymbol{z}\|^2}{2} \pm \frac{1}{2}\sqrt{(\|\boldsymbol{y}\|^2 - \alpha_\mathrm{s}\|\boldsymbol{z}\|^2)^2 + 4\alpha_\mathrm{s}|\boldsymbol{z}^\dagger\boldsymbol{y}|^2}$$

Specifically the GLR is

$$\mathrm{GLR} = \exp\left(\frac{\lambda_1^{(1)} - \|\boldsymbol{y}\|^2}{\sigma^2}\right)$$

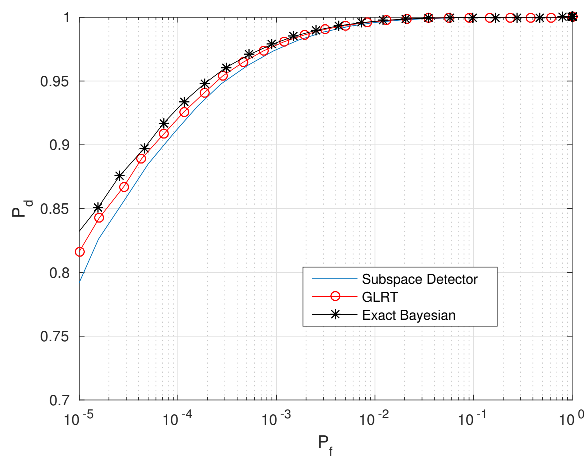

Fig. 1: Receiver operating characteristic curves for a scenario with $K = 1$ reference channel and one surveillance channel. In this example, the aggregate reference-channel SNR is $\mathrm{SNR_r} = -5\mathrm{dB}$ and the surveillance-channel SNR is $\mathrm{SNR_s} = -15\mathrm{dB}$. Other parameters are $N = 2^{11}$, $\beta_s^2 = 100$.

while the Bayesian marginalized likelihood ratio is

$$\frac{p(\boldsymbol{y},\boldsymbol{z}|H_1)}{p(\boldsymbol{y},\boldsymbol{z}|H_0)} = \frac{{}_1F_1(1;N;\lambda_1^{(\alpha_s)}/\sigma^2) - {}_1F_1(1;N;\lambda_2^{(\alpha_s)}/\sigma^2)}{(1+\beta_s^2)^2(\lambda_1^{(\alpha_s)} - \lambda_2^{(\alpha_s)})_1F_1(1;N;\|\boldsymbol{y}\|^2/\sigma^2)}$$

where ${}_1F_1(a;b;x)$ is the confluent hypergeometric function defined in [6].

The conventional detection method used for bistatic passive radar is to use cross-correlation of the direct path signal with the surveillance channel signal; i.e.,

$$\mathrm{SSD} = \boldsymbol{z}^\dagger P_{\boldsymbol{y}} \boldsymbol{z} = \frac{|\boldsymbol{y}^\dagger \boldsymbol{z}|^2}{\|y\|^2}$$

where $P_{\boldsymbol{y}}$ is the orthogonal projection on to the one-dimensional subspace spanned by $\boldsymbol{y}$. As observed in Section VI the performance of the three detectors has the Bayesian detector performing best and the SSD detector worst. However, the performance of the three detectors are not too different, with the difference decreasing as the SNR of the direct path signal increases.

We now consider the simplest case for multistatic passive radar, that of using two transmitters of opportunity with a single surveillance receiver. We will see in this scenario that the relative performances of the three basic detectors are very different.

V. Two Transmitters and One Surveillance Channel

Using recently derived results in [3] regarding integration over $G_{K,N}$, it is now possible to compute the marginalization integrals in (1) exactly. Specifically, using the notation

$$|A_{ij}|_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$

we have, for $K = 2$,

$$\int_{G_{2,N}} \exp(t\mathrm{Tr}(PW))d\mu(P) = \left| C_{ij} \sum_{\ell=1}^{M} [L(\boldsymbol{\lambda})^{-1}]_{i+M',\ell} \lambda_\ell^{i+M-3} {}_1F_1(1;N'+i-j;t\lambda_\ell) \right|_2$$

where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_M$ are the non-zero eigenvalues of the $N \times N$ hermitian matrix W, $N' = N-1$, $M' = M-2$,

$$C = \begin{pmatrix} 1 & N-2 \\ 1 & N-1 \end{pmatrix},$$

and

$$[L(\boldsymbol{\lambda})^{-1}]_{ij} = \begin{cases} 1 & \text{if } i=j=1 \\ 0 & \text{if } i>j \\ \prod_{\substack{\ell=1 \\ \ell\neq j}}^{i} \frac{1}{(\lambda_j-\lambda_\ell)} & \text{otherwise.} \end{cases}$$

Consider (1), for $M = 3$ and $K = 2$. In this case the data Z from the surveillance channel consists of a single data vector which we write as $\boldsymbol{z}$. The matrix $Y^\dagger Y + \alpha_s \boldsymbol{z}^\dagger \boldsymbol{z}$ has the same non-zero eigenvalues as the Gram matrix

$$G = \begin{pmatrix} YY^\dagger & \sqrt{\alpha_s}\boldsymbol{z}Y^\dagger \\ \sqrt{\alpha_s}Y\boldsymbol{z}^\dagger & \alpha_s\|\boldsymbol{z}\|^2 \end{pmatrix} \quad (2)$$

Denote these three non-zero eigenvalues by $\mu_1^{(\alpha_s)} \le \mu_2^{(\alpha_s)} \le \mu_3^{(\alpha_s)}$ and $Y^\dagger Y$ has two non-zero eigenvalues, $\rho_1 \le \rho_2$. The exact Bayesian likelihood ratio (1) becomes

$$\frac{p(X|H_1)}{p(X|H_0)} = \frac{\left| C_{ij} \sum_{\ell=1}^{3} [L(\boldsymbol{\mu}^{(\alpha_s)})^{-1}]_{i+1,\ell} \mu^{(\alpha_s)}{}_\ell^i {}_1F_1(1;N-1+i-j;\frac{\mu_\ell^{(\alpha_s)}}{\sigma^2}) \right|_2}{(1+\beta_s^2)^2 \left| C_{ij} \sum_{\ell=1}^{2} [L(\boldsymbol{\rho})^{-1}]_{i,\ell} \rho_\ell^{i-1} {}_1F_1(1;N-1+i-j;\frac{\rho_\ell}{\sigma^2}) \right|_2}$$

This is the particular case used in Section VI to obtain operating characteristic curves using simulated receiver data.

The performance of the exact Bayesian test is to be compared with that of the GLRT based on GLR statistic from [5]. That is, for known σ^2,

$$\mathrm{GLR} = \exp\left(\frac{1}{\sigma^2}\sum_{i=1}^{2}(\mu_i - \rho_i)\right) \quad (3)$$

where, $\mu_1 \ge \mu_2 \ge \mu_3$ are the non-zero eigenvalues of $Y^\dagger Y + \boldsymbol{z}^\dagger \boldsymbol{z}$, or equivalently, the eigenvalues of (2) with $\alpha_s = 1$. The quantities $\rho_1 \ge \rho_2$ are the non-zero eigenvalues of $Y^\dagger Y$, or equivalently, the eigenvalues of the 2×2 matrix $YY^\dagger$.

As in the bistatic case, the Bayesian detector is also compared to the multichannel generalization of the cross correlation detector, which we now refer to as the Subspace detector. This detector is derived by constructing the projection onto the subspace spanned by the columns of Y, that is,

$$P_Y = Y^\dagger (YY^\dagger)^{-1} Y.$$

The corresponding detection statistic is

$$\mathrm{SSD} = \boldsymbol{z}^\dagger P_Y \boldsymbol{z}. \quad (4)$$

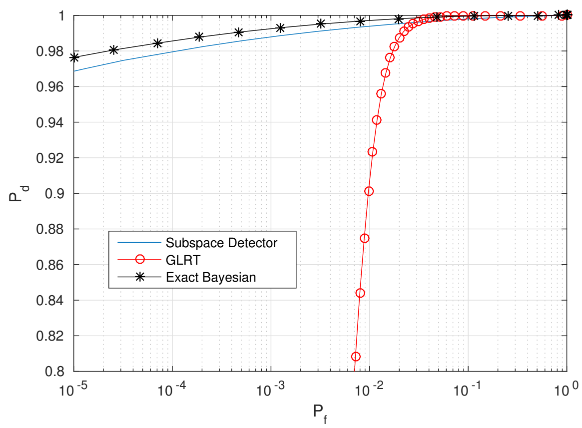

Fig. 2: Receiver operating characteristic curves for a scenario with $K = 2$ reference channels and one surveillance channel. In this example, the aggregate reference-channel SNR is $\mathrm{SNR_r} = 0\mathrm{dB}$ and the surveillance-channel SNR is $\mathrm{SNR_s} = -15\mathrm{dB}$. Other parameters are $N = 2^{12}$, $\beta_s^2 = 100$.

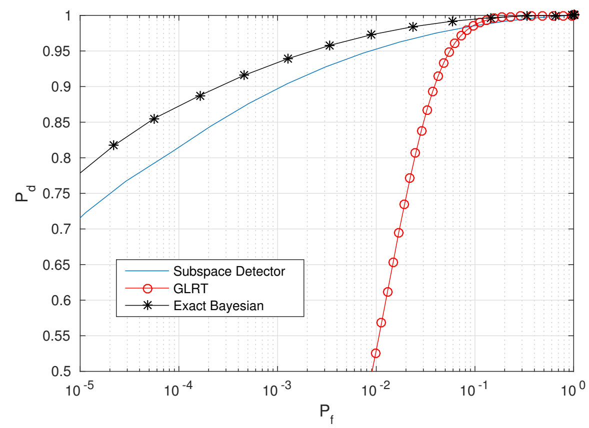

Fig. 3: Receiver operating characteristic curves for a scenario with $K = 2$ reference channels and one surveillance channel. In this example, the aggregate reference-channel SNR is $\mathrm{SNR_r} = -5\mathrm{dB}$ and the surveillance-channel SNR is $\mathrm{SNR_s} = -15\mathrm{dB}$. Other parameters are $N = 2^{12}$, $\beta_s^2 = 100$.

VI. Performance

The performances of the GLRT, Subspace and Bayesian detectors described in Sections IV and III, are evaluated by simulation. In the bistatic case there are a total of two receive antennas, one for the direct path and one surveillance receiver. The receiver noise level σ^2 is assumed to be known and the same across all receivers. The amplitude of the direct-path signal is determined by the SNR, with a random phase which is uniformly distributed. Figure 1 shows the detection performance of the three detectors in Section IV for a $-5\mathrm{dB}$ SNR direct path signal of length 2^{11} samples. The signal used is a binary phase coded waveform with a randomly selected code. The Bayesian detector consistently outperforms the other two. In this bistatic case the difference in performance is significant but not extreme, with the difference decreasing as the SNR of the direct path increases.

In the case of two transmitters and one surveillance channel, a total of three receive antennas ($M = 3$) are considered. Two of these receive direct-path signals from two distinct illuminators ($K = 2$). The third antenna is the surveillance-channel antenna, configured to collect signals scattered from a potential target. The receiver noise level σ^2 is assumed to be known and the same across all receivers. The amplitudes of the two direct-path signals are uniformly distributed with mean zero and same variance, and therefore the power in each path will be different. The total SNR is set to be 0 and $-5\mathrm{dB}$. The number of samples N is 2^{12}. The SNR of the received signal reflected from a target is $-15\mathrm{dB}$ for both cases. The performance of the exact Bayesian test is compared to that of the GLRT (3) and the Subspace detector (4).

Figure 2 shows the detection performance of the three detectors in Section III for a $0\mathrm{dB}$ SNR direct path signal. Figure 3 shows the corresponding results for the weaker $-5\mathrm{dB}$ SNR direct path signals. It can be seen that in this scenario that both the Bayesian detector and the Subspace detector have an overwhelming advantage over the GLRT detectors as the SNR of the direct path signals becomes progressively weaker. This is particularly true for lower false alarm rates. In comparison to the Bayesian detector, the Subspace detector has somewhat worse performance with the difference in performance increasing as the SNR of the direct path approaches that of the target.

VII. Conclusion

In this paper exact Bayesian detectors for passive radar detection have been derived using non-informative prior distributions. Exact Bayesian detectors were presented for both the conventional bistatic configuration and a configuration involving two direct path antennas collecting two emitters of opportunity and a single surveillance antenna (tristatic). For these configurations the Bayesian detector consistently outperforms both the GLRT and the Subspace detector, while in the tristatic configuration with weak direct path signals the Bayesian detector overwhelming outperforms the GLRT.

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