THE GEOMETRY OF INVARIANTS FOR GENERALIZED COHERENCE TESTS

Stephen D. Howard[1] *Douglas Cochran*[2] *Songsri Sirianunpiboon*[1]

[1]Defence Science and Technology Group
PO Box 1500, Edinburgh 5111, Australia

[2]School of Mathematical and Statistical Sciences
Arizona State University, Tempe AZ 85287-5706 USA

ABSTRACT

This paper considers the problem of testing for mutual independence of multiple sets of complex Gaussian vectors. This problem has classical roots in statistics and has been of recent interest in the signal processing literature in connection with multi-channel signal detection. The maximal invariant statistic for this problem is described both as a collection of subspaces of the data space (i.e., points on a complex Grassmannian manifold) and as a corresponding set of complex matrices. The distribution of the maximal invariant is also derived under both hypotheses in the testing problem.

***Index Terms*—** Multiple-channel detection, Invariance, Maximal invariant, Wijsman's Theorem

1. INTRODUCTION

Invariances of various types feature prominently in the history of statistical hypothesis testing. In particular, the problem of testing for independence among multiple sets of normal random variables is classical in the statistical literature [1] and is fundamental in modern multi-channel signal detection. This circle of detection problems was studied in the 1970s in connection with in passive sonar applications involving two channels of sensor data (e.g., [2]) and somewhat later with multiple and generalized coherence tests for multiple channels [3, 4, 5]. There has been a recent resurgence of interest in problems of this type, much of it motivated by passive radar applications (e.g., [6, 7, 8, 9, 10]) and spectrum sensing for cognitive radio (e.g., [11, 12, 13]). A space-time perspective was introduced in [14] that generalized this circle of problems and associated statistical methods. Invariances are a recurring theme in this work (e.g., [15, 16, 17, 18, 19, 20]).

This paper describes the maximal invariant statistic for a fairly general case of the type of multi-channel detection problems treated in the references cited above. In Section 3, this statistic is described equivalently as a collection of subspaces of the data space, a set of points on a complex Grassmannian manifold, and as a set of complex matrices. Section 4 derives the distributions of the maximal invariant derived directly under both hypotheses of the detection problem, and Section 5 compares this derivation to one that uses Wijsman's theorem.

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2. MODEL AND PROBLEM FORMULATION

Consider the following problem. There are M sets, each containing K zero-mean Gaussian random vectors in $\mathbb{C}^N$. Within each random vector the components are independently distributed. Arranging the vectors $\{\boldsymbol{x}_{m,1},\ldots,\boldsymbol{x}_{m,K}\}$ as the columns of the $N\times K$ matrix X_m, the probability density of the m^{th} set of vectors is taken as

$$p(X_m|R_m)=\frac{1}{(2\pi)^{KN}\det\left(R_m\right)^N}\exp\left(-\frac{1}{2}\mathrm{Tr}\left(X_mR_m^{-1}X_m^\dagger\right)\right)$$

For each m, take the data matrix X_m to belong to the space $\mathcal{X}$ of $N\times K$ matrices with two properties: (i) they have full rank K, and (ii) the $K\times K$ sub-matrix consisting of the first K rows of X is invertible. Note that, with the density assumed distribution, the matrices X_m have these properties with probability one. The reasons for these conditions will become clear in Section 3.

The question to be addressed is whether the M sets of vectors are mutually independent. In order to formulate the associated hypothesis test succinctly, denote the space of $L\times L$ positive definite matrices by $\mathcal{P}_L$, and further denote by $\mathcal{P}_L^M$ the space of $LM\times LM$ block-diagonal positive definite matrices with $L\times L$ blocks. Define a $N\times MK$ matrix $\boldsymbol{X}=(X_1,\ldots,X_M)$ and note that the density of $\boldsymbol{X}$ is

$$\begin{aligned}p(\boldsymbol{X}|\boldsymbol{R})&=\frac{1}{(2\pi)^{KNM}\det\left(\boldsymbol{R}\right)^N}\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{X}\boldsymbol{R}^{-1}\boldsymbol{X}^\dagger\right)\right)\\&=\frac{1}{(2\pi)^{KNM}\det\left(\boldsymbol{R}\right)^N}\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{R}^{-1}\boldsymbol{X}^\dagger\boldsymbol{X}\right)\right)\end{aligned}\tag{1}$$

for $\boldsymbol{R}\in\mathcal{P}_{MK}$. The hypothesis test can now be stated as H_1 against H_0 with

$$H_1:\ \boldsymbol{X}\text{ is distributed according (1) for some }\boldsymbol{R}\in\mathcal{P}_{MK}$$
$$H_0:\ \boldsymbol{X}\text{ is distributed according (1) for some }\boldsymbol{R}\in\mathcal{P}_K^M$$

It is observed in [19] that this detection problem is invariant under the following group actions. First take $\{S_1,\cdots,S_M\}$ to be a set matrices with each $S_m\in\mathrm{GL}(K,\mathbb{C})$, the group of $K\times K$ invertible complex matrices. In block form $\boldsymbol{S}\in\mathrm{GL}(K,\mathbb{C})^M$ is the invertible block diagonal matrix with blocks $\{S_1,\cdots,S_M\}$. Consider the (right) action of $\mathrm{GL}(K,\mathbb{C})^M$ on $\boldsymbol{X}\in\mathbb{C}^{N\times MK}$ given by

$$\boldsymbol{X}\to\boldsymbol{X}\boldsymbol{S}$$

and the corresponding action of $\mathrm{GL}(K,\mathbb{C})^M$ on $\boldsymbol{R}\in\mathcal{P}_{KM}$

$$\boldsymbol{R}\to\boldsymbol{S}^\dagger\boldsymbol{R}\boldsymbol{S}.\tag{2}$$

The action (2) preserves the block-diagonal structure of the matrices in $\mathcal{P}_K^M \subset \mathcal{P}_{KM}$. Under the combined group action it is easily seen that

$$p(\boldsymbol{XS}|\boldsymbol{S}^\dagger \boldsymbol{RS}) = p(\boldsymbol{X}|\boldsymbol{R})$$

if $\boldsymbol{R}$ structured either as in H_0 or as in H_1. Additionally, the group S_M of permutations that permute the order of the M sets of vectors also defines an invariance of the problem. If P_σ is an $M \times M$ permutation matrix corresponding to the permutation $\sigma \in \mathrm{S}_M$, then its respective actions on $\mathcal{X}^M$ and $\mathcal{P}_{KM}$ are given by

$$\boldsymbol{X} \to \boldsymbol{P}_\sigma \boldsymbol{X}$$
$$\boldsymbol{R} \to \boldsymbol{P}_\sigma^\dagger \boldsymbol{R} \boldsymbol{P}_\sigma,$$

where $\boldsymbol{P}_\sigma = P_\sigma \otimes I$, with I the $K \times K$ identity matrix. The full group of invariances is generated by the composition of the above actions of $\mathrm{GL}(K, \mathbb{C})^M$ and S_M and will be denoted by F.

Another symmetry of the problem involves the unitary group $\mathrm{U}(N)$ with action on $\mathcal{X}^M$ by

$$\boldsymbol{X} \to \boldsymbol{UX}.$$

This action leaves $p(\boldsymbol{X}|\boldsymbol{R})$ invariant with the identity action on the parameter $\boldsymbol{R}$ and is consequently unimportant in the context of constructing invariant tests. In fact, this invariance is a consequence of the sufficiency of the statistic $\boldsymbol{XX}^\dagger$ for $\boldsymbol{X}$. The approach taken here is to first the maximal invariants under the group action by F only and then to further refine the maximal invariants to include the group action by $U(N)$.

3. MAXIMAL INVARIANTS

3.1. Maximal invariants under the action of F

The maximal invariants of the group action of F on $\mathcal{X}^M$ can be determined in the following way. Begin with the data $X \in \mathcal{X}$ from one of the M sets. Applying all of the elements of $\mathrm{GL}(K, \mathbb{C})$ to X generates a set of matrices in $\mathcal{X}$, the orbit $\mathcal{O}_X$ of X under the action of $\mathrm{GL}(K, \mathbb{C})$; i.e.,

$$\mathcal{O}_X = \{XS | S \in \mathrm{GL}(K, \mathbb{C})\}.$$

The space $\mathcal{X}$ decomposes completely into disjoint orbits; i.e., every $X \in \mathcal{X}$ belongs to exactly one orbit. A matrix $X \in \mathcal{X}$ consists of K linearly independent columns, since it was assumed to have full rank. The linear span of the columns of X is thus a K-dimensional subspace of $\mathbb{C}^N$ denoted by $\langle X \rangle$. Applying $S \in \mathrm{GL}(K, \mathbb{C})$ to X creates a matrix $X' = XS$ with columns that span exactly the same subspace; i.e., $\langle X \rangle = \langle XS \rangle$. Furthermore, if $\langle X \rangle = \langle X' \rangle$ there is a unique $S \in \mathrm{GL}(K, \mathbb{C})$ such that $X' = XS$. This all implies that every element X of a single orbit generates the same K-dimensional subspace of $\mathbb{C}^N$ and elements on different orbits generate distinct subspaces. Thus, the orbits of the action of $\mathrm{GL}(K, \mathbb{C})$ on $\mathcal{X}$ are in one-to-one correspondence with, and can thus be uniquely labeled by, the set of all K-dimensional subspaces of $\mathbb{C}^N$; i.e., the Grassmannian $G_K(\mathbb{C}^N)$, which is a smooth complex manifold of complex dimension $K(N-K)$. This somewhat abstract formulation will be valuable in the development to follow, although a more concrete setup will also be described below.

Consider how the framework just introduced extends to the full data space $\mathcal{X}^M$. Every element of F can be written as a product $\boldsymbol{SP}_\sigma$ for some $\boldsymbol{S} \in \mathrm{GL}(K, \mathbb{C})^M$ and $\sigma \in \mathrm{S}_M$. Consequently, the orbits of the action of F on $\mathcal{X}^M$ are

$$\mathcal{O}_{\boldsymbol{X}} = \{\boldsymbol{XSP}_\sigma \mid \boldsymbol{S} \in \mathrm{GL}(K, \mathbb{C})^M, \sigma \in \mathrm{S}_M\}$$

To understand the orbits, first consider just the $\mathrm{GL}(K, \mathbb{C})^M$ action on $\mathcal{X}^M$. The elements of $\mathcal{X}^M$ consist of vectors $(X_1, \ldots, X_M)$ of elements in $\mathcal{X}$, so it can be seen that the $\mathrm{GL}(K, \mathbb{C})^M$ orbits can be identified with vectors of K-dimensional subspaces $(\langle X_1 \rangle, \ldots, \langle X_M \rangle)$. Now the action of S_M on $\mathcal{X}^M$ are the union of the $M!$ $\mathrm{GL}(K, \mathbb{C})^M$ orbits labeled by $(\langle X_{\sigma(1)} \rangle, \ldots, \langle X_{\sigma(M)} \rangle)$, for $\sigma \in \mathrm{S}_M$. Notice that it is only necessary to consider the group action of S_M induced on the $\mathrm{GL}(K, \mathbb{C})^M$ orbits rather than on the whole of $\mathcal{X}^M$. Thus, the orbits in $\mathcal{X}^M$ under the action of F are labeled by sets of M, K-dimensional subspaces of $\mathbb{C}^N$; i.e., there is no inherent ordering of the subspaces. Equivalently, the orbits can be considered to be labeled by sets of M points on the Grassmannian $G_K(\mathbb{C}^N)$ (Technically, due to the definition of $\mathcal{X}$ above, the points lie in an open subset of the Grassmannian. Under the measure induced by probability distribution of $\boldsymbol{X}$, however, this subset covers the entire Grassmannian except for a set of measure zero. If necessary, this open subset will be referred to as $\mathcal{U}(\mathcal{X})$). *The maximal invariant statistic for this problem is given by the unordered set* $\{\langle X_1 \rangle, \ldots, \langle X_M \rangle\}$ *of K-dimensional subspaces in* $\mathbb{C}^N$.

This analysis can can be make more concrete by finding a way to assign a unique basis to each subspace in a smooth way. Let $X \in \mathcal{X}$ be any basis for the K-dimensional subspace $V = \langle X \rangle$ and write

$$X = \begin{pmatrix} S \\ Y \end{pmatrix}$$

where $S \in \mathbb{C}^{K \times K}$ and $Y \in \mathbb{C}^{(N-K) \times K}$. By definition of the space $\mathcal{X}$, S is invertible; i.e., $S \in \mathrm{GL}(K, \mathbb{C})$. The basis X can now be put into the standard form by applying S^{-1}

$$X' = \begin{pmatrix} I_K \\ Z \end{pmatrix}$$

where $Z = YS^{-1}$ and then orthonormalized to give

$$X(Z) = \begin{pmatrix} (I_K + Z^\dagger Z)^{-1/2} \\ Z(I_K + Z^\dagger Z)^{-1/2} \end{pmatrix}. \tag{3}$$

Thus, for each matrix $Z \in \mathbb{C}^{(N-K) \times K}$ there is a unique subspace $V_Z = \langle X(Z) \rangle$, and for each K-dimensional subspace of $\mathbb{C}^N$ (in $\mathcal{U}(\mathcal{X})$) there is a unique matrix Z. Thus, the desired concrete characterization of the maximal invariant for this problem is: *The maximal invariant statistic for this problem is given by the set* $\{Z(X_1), \ldots, Z(X_M)\}$ *of matrices in* $\mathbb{C}^{(N-K) \times K}$.

As a compact notation, for an ordered collection of matrices in $\mathbb{C}^{(N-K) \times K}$, $\boldsymbol{Z} = (Z_1, \ldots, Z_M)$, the notation $\boldsymbol{X}(\boldsymbol{Z})$ stands for the matrix $(X(Z_1), \ldots, X(Z_M))$.

3.2. Maximal invariants under the action of $U(N) \times F$

Having determined the maximal invariant for the action of F, the maximal invariant under the full group action by $U(N) \times F$ can be found by considering the induced group action of $U(N)$ on the collection of subspaces $\{\langle X_1 \rangle, \ldots, \langle X_M \rangle\} \in G_K(\mathbb{C}^N)^M$. From this point of view, the orbits correspond to the collective motion of M points (subspaces) in $G_K(\mathbb{C}^N)$ under the action of $U(N)$. Thus, determination of the maximal invariant corresponds to putting the subspaces into some standard configuration under the action of $U(N)$. Because the action of $U(N)$ on $G_K(\mathbb{C}^N)$ is transitive, one subspace in the collection can always be moved to the subspace spanned by the first K standard basis vectors in $\mathbb{C}^N$.

In determining a concrete representation of the maximal invariant, it is more convenient to start with the action of $U(N) \times F$ on

$\mathcal{X}^M$. For $\boldsymbol{X} \in \mathcal{X}^M$ there is a unique pair of matrices T and Q_1, where T is an $MK \times MK$ upper triangular matrix with positive real diagonal entries and Q_1 is a $N \times MK$ matrix with $Q_1^\dagger Q_1 = I_K$, such that

$$\begin{pmatrix} Q_1^\dagger \\ Q_2^\dagger \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} T \\ 0_{(N-MK)\times MK} \end{pmatrix}$$

Here Q_2 is any matrix which ensures (Q_1, Q_2) is unitary.

The matrix T can be written, in terms of $K \times K$ blocks, as

$$T = \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_M \end{pmatrix} \begin{pmatrix} I & I & I & \cdots \\ 0 & U_2^\dagger T_{22} T_{12}^{-1} U_1 & U_2^\dagger T_{23} T_{13}^{-1} U_1 & \cdots \\ 0 & 0 & U_3^\dagger T_{33} T_{13}^{-1} U_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$\times \begin{pmatrix} U_1^\dagger T_{11} & & \\ & \ddots & \\ & & U_1^\dagger T_{MM} \end{pmatrix}$$

Now, choose $U_1, \ldots, U_M$ to be such that $U_2^\dagger T_{22} T_{12}^{-1} U_1$ is diagonal with real positive entries (SVD). This only specifies U_1 and U_2 up to multiplication by a common diagonal unitary matrix. The extra degrees of freedom are used to make the first column of $U_2^\dagger T_{23} T_{13}^{-1} U_1$ real and positive. The remaining $U_3, \ldots, U_M$ are chosen to make $U_j^\dagger T_{jj} T_{1j}^{-1} U_1$ upper triangular with real positive diagonal, for $j = 3, \ldots, M$. Thus, the maximal invariant for the $U(N) \times \mathrm{GL}(K, \mathbb{C})^M$ action on $\mathcal{X}^M$ can be specified by an $MK \times (M-1)K$ matrix $\hat{\boldsymbol{Z}}$ of the form (shown for $K = 3$):

$$\hat{\boldsymbol{Z}} = \begin{pmatrix} r & 0 & 0 & r & x & x & x & x & x & \cdots \\ 0 & r & 0 & r & x & x & x & x & x & \cdots \\ 0 & 0 & r & r & x & x & x & x & x & \cdots \\ 0 & 0 & 0 & r & x & x & x & x & x & \cdots \\ 0 & 0 & 0 & 0 & r & x & x & x & x & \cdots \\ 0 & 0 & 0 & 0 & 0 & r & x & x & x & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & r & x & x & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & x & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the element positions labeled with r contain arbitrary real positive numbers and those labeled with x contain arbitrary complex numbers. Finally, the action of S_M on $\mathcal{X}^M$ implies that $\hat{\boldsymbol{Z}}$ matrices derived from different permutations of $X_1, \ldots, X_M$ are equivalent. This equivalence class, which contains $M!$ elements, will be denoted by $[\hat{\boldsymbol{Z}}]$. The individual elements of this equivalence class will be denoted by $\hat{\boldsymbol{Z}}_\sigma$, for $\sigma \in S_M$. This quantity is the maximal invariant under the action of $U(N) \times F$ on $\mathcal{X}^M$.

Writing $\hat{\boldsymbol{Z}} = (\hat{Z}_2, \ldots, \hat{Z}_M)$ and $\hat{Z}_1 = 0$, an element of $\mathcal{X}$ can be defined as

$$\boldsymbol{X}(\hat{\boldsymbol{Z}}) = \left(X(0_{N,K}), X\begin{pmatrix} \hat{Z}_2 \\ 0_{N-MK,K} \end{pmatrix}, \ldots, X\begin{pmatrix} \hat{Z}_M \\ 0_{N-MK,K} \end{pmatrix} \right)$$

where $X(\cdot)$ denotes the mapping defined in (3). Correspondingly $\boldsymbol{G}(\hat{\boldsymbol{Z}})$ is defined as

$$\boldsymbol{G}(\hat{\boldsymbol{Z}}) = \boldsymbol{X}(\hat{\boldsymbol{Z}})^\dagger \boldsymbol{X}(\hat{\boldsymbol{Z}}).$$

Finally, it is interesting to realize that the maximal invariant for this hypothesis test under the action of the full invariance group $U(N) \times F$ is, in fact, a sufficient statistic for the maximal invariant $\{Z_1, \ldots, Z_M\}$ under the action of F alone.

4. DISTRIBUTIONS OF THE MAXIMAL INVARIANT UNDER F

To exploit invariance of a statistical problem it is important to compute the distribution of the maximal invariant under the various hypotheses in the problem. The direct method involves changing coordinates so that the maximal invariant is made explicit and then marginalizing the data distribution over the remaining coordinates. For binary hypothesis testing, an alternate approach is the use of Wijsman's theorem[21, 22], through which one can obtain the ratio of the densities of the maximal invariant under H_1 and H_0 by averaging the full densities over orbits in the data space under the action of the invariance group. In fact, Wijsman's theorem has been used recently by Ramírez et al. [19] to derive the locally most powerful invariant test (LMPIT) for the problem addressed in this paper. What might be seen as an advantage is that Wijsman's theorem can be applied without determining the nature of the maximal invariant. In this section, the distribution for the maximal invariant for action of F on $\mathcal{X}^M$ is computed directly by marginalization and the results are then contrasted with those obtained from Wijsman's theorem in Section 5.

The first task in computing the distribution of the maximal invariant is parameterization of $\mathcal{X}^M$ in terms of the invariant. A point in $\mathcal{X}^M$ is specified by $(\boldsymbol{S}, \{Z_1, \ldots, Z_M\}, \sigma)$ where $\boldsymbol{S} \in \mathrm{GL}(K, \mathbb{C})^M$, $\{Z_1, \ldots, Z_M\}$ is a set of M matrices in $\mathbb{C}^{N-K()\times K}$ and $\sigma \in \mathrm{S}_M$. Denoting $\boldsymbol{Z}_\sigma = (Z_{\sigma(1)}, \ldots, Z_{\sigma(M)})$, bZ_σ relates to the original parameterization through

$$\begin{aligned} \boldsymbol{X} &= \boldsymbol{X}(\boldsymbol{Z}_\sigma)\boldsymbol{S} \\ &= \boldsymbol{X}(\boldsymbol{Z})\boldsymbol{P}_\sigma \boldsymbol{S} \end{aligned} \tag{4}$$

where $\boldsymbol{X}(\boldsymbol{Z})$ is the mapping defined in Section 3.1. The next step is to determine the Jacobian for this change of variables. For a given $Z \in \mathbb{C}^{N-K()\times K}$, the $N \times (N-K)$ matrix defined by

$$Y(Z) = \begin{pmatrix} -Z^\dagger (I_{N-K} + ZZ^\dagger)^{-1/2} \\ (I_{N-K} + ZZ^\dagger)^{-1/2} \end{pmatrix} \tag{5}$$

has the property that

$$Y(Z)^\dagger X(Z) = 0.$$

In fact, $Y(Z)$ is the standard basis (in the sense described in Section 3.1) for the orthogonal complement of the subspace $\langle X(Z) \rangle$; i.e., $\langle Y(Z) \rangle = \langle X(Z) \rangle^\perp$. From (4),

$$\underset{\cdot}{\mathrm{X}}_m S_m^{-1} = \underset{\cdot}{\mathrm{X}}(Z) + X(Z) \underset{\cdot}{\mathrm{S}}_m S_m^{-1}.$$

Multiplying from the left by the unitary matrix $\begin{pmatrix} Y(Z)^\dagger \\ X(Z)^\dagger \end{pmatrix}$ gives

$$\begin{pmatrix} Y(Z)^\dagger \\ X(Z)^\dagger \end{pmatrix} \underset{\cdot}{\mathrm{X}}_m S_m^{-1} = \begin{pmatrix} Y(Z)^\dagger \underset{\cdot}{\mathrm{X}}(Z) \\ X(Z)^\dagger \underset{\cdot}{\mathrm{X}}(Z) \end{pmatrix} + \begin{pmatrix} 0 \\ dS_m S_m^{-1} \end{pmatrix}.$$

Using the methods described in [23], this implies that

$$|S_m|^{-2N} \prod_{nj} d\,\mathrm{Re}\,(X_{nj})\, d\,\mathrm{Im}\,(X_{nj}) = d\mu_{\mathrm{GL}(K,\mathbb{C})}(S_m)\, d\mu_{G_K(\mathbb{C}^N)}(Z)$$

where $d\mu_{\mathrm{GL}(K,\mathbb{C})}(S_m)$ is the Haar measure on $\mathrm{GL}(K, \mathbb{C})$,

$$d\mu_{\mathrm{GL}(K,\mathbb{C})}(S) = |S|^{-2K} \prod_{ij} d\,\mathrm{Re}\,(S_{ij})\, d\,\mathrm{Im}\,(S_{ij}),$$

and $d\mu_{G_K(\mathbb{C}^N)}(Z)$ is the invariant measure on the Grassmannian $G_K(\mathbb{C}^N)$ in the coordinates Z. Now,

$$Y(Z)^\dagger \underset{\cdot}{X}(Z) = (I_{N-K} + ZZ^\dagger)^{-1/2} \underset{\cdot}{Z}(I_K + Z^\dagger Z)^{-1/2}$$

and so

$$d\mu_{G_K(\mathbb{C}^N)}(Z) = \left|I_K + Z^\dagger Z\right|^{-N} \prod_{ij} d\,\mathrm{Re}\,(Z_{ij})\, d\,\mathrm{Im}\,(Z_{ij}).$$

In total, the Lebesgue measure on $\mathcal{X}^M$ is

$$d\boldsymbol{X} = |\boldsymbol{S}|^{2(N-K)} d\boldsymbol{S} \prod_m \left|I_K + Z_m^\dagger Z_m\right|^{-N} dZ_m.$$

In what follows, it is convenient to introduce the notation

$$\boldsymbol{G}(\boldsymbol{Z}_\sigma) = \boldsymbol{X}(\boldsymbol{Z}_\sigma)^\dagger \boldsymbol{X}(\boldsymbol{Z}_\sigma),$$

and note that the diagonal blocks of $\boldsymbol{G}(\boldsymbol{Z}_\sigma)$ are

$$X(Z_m)^\dagger X(Z_m) = I_K. \tag{6}$$

All subsequent densities in this section are with respect to the invariant measure, $\prod_m d\mu_{G_K(\mathbb{C}^N)}(Z_m)$, on $G_K(\mathbb{C}^N)^M$. The density of the maximal invariant can now be computed under H_0 as

$$\begin{aligned} &p(\{Z_1, \ldots, Z_m\}|\boldsymbol{R} \in \mathcal{P}_K^M) \\ &= \sum_{\sigma \in S_M} \int_{\mathrm{GL}(K,\mathbb{C})^M} \frac{\left|\boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^\dagger\right|^N e^{-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^\dagger \boldsymbol{G}(\boldsymbol{Z}_\sigma)\right)}}{(2\pi)^{KNM}} \frac{d\boldsymbol{S}}{|\boldsymbol{S}|^{2K}}. \end{aligned}$$

The covariance $\boldsymbol{R} \in \mathcal{P}_K^M$, so the change of variable $\boldsymbol{S} \to \boldsymbol{S}\boldsymbol{R}^{-1/2}$ leaves $\boldsymbol{S} \in \mathrm{GL}(K,\mathbb{C})^M$. In addition, the diagonal blocks of $\boldsymbol{G}(\boldsymbol{Z}_\sigma)$ are I_k. This implies

$$\begin{aligned} &\int_{\mathrm{GL}(K,\mathbb{C})^M} \frac{\left|\boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^\dagger\right|^N \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^\dagger \boldsymbol{G}(\boldsymbol{Z}_\sigma)\right)\right)}{(2\pi)^{KNM}} \frac{d\boldsymbol{S}}{|\boldsymbol{S}|^{2K}} \\ &= \int_{\mathrm{GL}(K,\mathbb{C})^M} \frac{\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{S}\boldsymbol{S}^\dagger\right)\right)}{(2\pi)^{KNM}} |\boldsymbol{S}|^{2(N-K)} d\boldsymbol{S} \end{aligned}$$

which is constant. Thus, the density of the maximal invariant under H_0 is

$$\begin{aligned} p(\{Z_1, \ldots, Z_m\}|\boldsymbol{R} \in \mathcal{P}_K^M) &= \frac{M!}{\mathrm{vol}(G_K(\mathbb{C}^N))^M} \\ &= M! \left(\frac{\prod_{\ell=N-K}^{N-1} \ell!}{(2\pi)^{K(N-K)} \prod_{\ell=0}^{K-1} \ell!}\right)^M \end{aligned} \tag{7}$$

That is, under H_0, the subspaces comprising the maximal invariant are iid and uniformly distributed on the Grassmannian $G_K(\mathbb{C}^N)$.

In a similar way, the density of the maximal invariant under H_1 is given by

$$\begin{aligned} &p(\{Z_1, \ldots, Z_m\}|\boldsymbol{R} \in \mathcal{P}_{KM}) \\ &= \sum_{\sigma \in S_M} \int_{\mathrm{GL}(K,\mathbb{C})^M} \frac{\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{R}_{\boldsymbol{S},\sigma}^{-1} \boldsymbol{G}(\boldsymbol{Z})\right)\right)}{(2\pi)^{KNM} |\boldsymbol{R}_{\boldsymbol{S},\sigma}|^N} \frac{d\boldsymbol{S}}{|\boldsymbol{S}|^{2K}} \end{aligned} \tag{8}$$

where $\boldsymbol{R}_{\boldsymbol{S},\sigma} = \boldsymbol{P}_\sigma^\dagger \boldsymbol{S}^\dagger \boldsymbol{R}\boldsymbol{S}\boldsymbol{P}_\sigma$.

5. DENSITY DERIVATION VIA WIJSMAN'S THEOREM

In this section, the results of the previous section are re-derived using techniques involved in the development of Wijsman's Theorem; i.e., obtaining marginal densities for maximal invariants by averaging the full density over the orbits of the invariance group action on the data space. The density for the maximal invariant for the full invariance group $U(N) \times F$ with respect to the induced measure on $G_K(\mathbb{C}^N)/U(N)$ is then computed.

Define the F-invariant measure on $\mathcal{X}^M$ by

$$d\mu_0(\boldsymbol{X}) = \prod_{m=1}^M \frac{dX_m}{|X_m^\dagger X_m|^N}$$

and note that $\mathrm{GL}(K,\mathbb{C})$ is unimodular [24]. The results of the last section imply

$$d\mu_0(\boldsymbol{X}) = \prod_{m=1}^M d\mu_{\mathrm{GL}(K,\mathbb{C})}(S_m) \times \prod_{m=1}^M d\mu_{G_K(\mathbb{C}^N)}(Z_m).$$

The density for $\boldsymbol{X}$ with respect to this measure is

$$f(\boldsymbol{X}|\boldsymbol{R}) = \left(\prod_{m=1}^M |X_m^\dagger X_m|^N\right) p(\boldsymbol{X}|\boldsymbol{R})$$

The results of Andersson [25] (cf. also [22, Theorem 5.5 & 5.7]) associated with his treatment of Wijsman's theorem thus imply that

$$\begin{aligned} &p(\{Z_1, \ldots, Z_M\}|\boldsymbol{R}) \\ &= \sum_{\sigma \in S_M} \int_{\mathrm{GL}(K,\mathbb{C})^M} f(\boldsymbol{X}(\boldsymbol{Z}_\sigma)\boldsymbol{S}|\boldsymbol{R}) \frac{d\boldsymbol{S}}{|\boldsymbol{S}|^{2K}} \\ &= \prod_{m=1}^M |X_m^\dagger X_m|^N \\ &\quad \times \sum_{\sigma \in S_M} \int_{\mathrm{GL}(K,\mathbb{C})^M} p(\boldsymbol{X}(\boldsymbol{Z}_\sigma)\boldsymbol{S}|\boldsymbol{R}) |\boldsymbol{S}|^{2(N-K)} d\boldsymbol{S} \end{aligned}$$

with respect $\prod_{m=1}^M d\mu_{G_K(\mathbb{C}^N)}(Z_m)$, the quotient measure on $G_K(\mathbb{C}^N)^M$. This is consistent with (7) and (8).

Finally, the probability distribution for the maximal invariant associated with the action of $U(N) \times F$ can be obtained starting with the distribution for the maximal invariant for F through the $U(N)$ group action of $G_K(\mathbb{C}^N)^M$. Denoting $\boldsymbol{R}_{\boldsymbol{S}} = \boldsymbol{S}^{-1}\boldsymbol{R}\boldsymbol{S}$, the result is

$$\begin{aligned} &p([\hat{\boldsymbol{Z}}]|\boldsymbol{R} \in \mathcal{P}_{KM}) \\ &= \mathrm{vol}(U(N)) \sum_{\sigma \in S_M} \int_{\mathrm{GL}(K,\mathbb{C})^M} \frac{\exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{R}_{\boldsymbol{S}}^{-1} \boldsymbol{G}(\hat{\boldsymbol{Z}}_\sigma)\right)\right)}{(2\pi)^{KNM} |\boldsymbol{R}_{\boldsymbol{S}}|^N} \frac{d\boldsymbol{S}}{|\boldsymbol{S}|^{2K}} \end{aligned}$$

with respect to the induced measure on $G_K(\mathbb{C}^N)^M/U(N)$. Under H_0 the distribution of the maximal invariant is uniform on $G_K(\mathbb{C}^N)^M/U(N) \times S_M$; i.e.,

$$p([\hat{\boldsymbol{Z}}]|\boldsymbol{R} \in \mathcal{P}_K^M) = \frac{M!\,\mathrm{vol}(U(N))}{\mathrm{vol}(G_K(\mathbb{C}))^M}$$

The exact form of the quotient measure on $G_K(\mathbb{C}^N)^M/U(N)$ will be considered in a future publication.

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