HIGHER-DIMENSIONAL COHERENCE OF SUBSPACES

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ABSTRACT

A measure of the smallest angle between two p-dimensional subspaces of a higher-dimensional vector space is introduced. This p-coherence is seen to generalize standard measures of coherence, and its relationship to canonical correlation is noted. The use of p-coherence as a test statistic for the presence of a common p-dimensional signal subspace in data collected by two arrays of $K > p$ sensors is proposed.

***Index Terms*—** Multi-channel detection; Coherence; Canonical correlation; Exterior algebra

1. INTRODUCTION

Geometric ideas have played important roles in statistical signal processing, particularly in connection with invariances arising from symmetries of probability distributions and with relationships between subspaces spanned by vectors in an ambient linear space. Geometric reasoning is often involved in deriving statistical tests and estimators corresponding to optimality or other standard criteria ((e.g., maximum likelihood). In fact, geometric insight has sometimes led to effective schemes for detection, estimation, and other typical statistical signal processing objectives in advance of the derivation of such tests. One such case is generalized coherence [1, 2], which was introduced in 1988 specifically in connection with the problem of detecting the presence of a common but unknown signal in noisy data collected at multiple sensors. The geometrical nature of the test statistic, essentially the determinant of a Gram matrix formed from the data vectors, provided heuristic justification for its use. And the availability of its distribution under specific H_0 assumptions allowed determination of thresholds corresponding to desired false alarm probabilities in Neyman-Pearson type testing. It was only sometime later that the GC statistic was identified as the solution to precisely formulated statistical testing problems [3].

This paper introduces a geometrically motivated statistic for problems in which it is desirable to determine from noisy data whether the respective spans of two collections of vectors in a high-dimensional space share a common subspace. Investigation of such situations is motivated by the increasing prevalence of MIMO architectures in RF sensing and communications systems. Consider, for example, a scenario in which a transmit array emits a rank-P signal which is received at two geographically separated receiver arrays. In addition to this rank-P signal, each receiver array receives signals from local sources that are too weak to have discernible effect on the other receiver array. To ascertain the presence of a common signal at the two receiver arrays and also its rank, which is a valuable signature of the emitter, a test for a common subspace in the channels of noisy received data from the two sensor arrays is warranted.

The work here contributes to an active circle of research on multi-sensor detection and estimation of signals in which rank is known or is to be estimated from data collected at multiple sensors; e.g., [4, 5, 6, 7, 8, 9].

2. MODEL AND PROBLEM FORMULATION

The approach introduced here to identify a common p-dimensional subspace in the spans of two sets of vectors is based on the following geometric formulation. Consider two sets of complex N-vectors, each set containing $K < N$ vectors. Denote these by $C_X = \{\mathbf{x}_1, \cdots, \mathbf{x}_K\}$ and $C_Y = \{\mathbf{y}_1, \cdots, \mathbf{y}_K\}$ and assume that each set has been separately orthonormalized. The question to be addressed is whether the respective spans V_X of C_X and V_Y of C_Y contain a common subspace of dimension $p \geq 1$.

Denote by G_X the $K \times K$ gram matrix for the set C_X; i.e.,

$$[G_X]_{nm} = \langle \mathbf{x}_n, \mathbf{x}_m \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^N$. Similarly, denote the gram matrix for C_Y by G_Y. Because of the orthonormality assumption, $G_X = G_Y = I_K$, the $K \times K$ identity matrix. The geometric relationship between C_X and C_Y is encoded in the $K \times K$ cross gram matrix $[H_{XY}]_{nm} = \langle \mathbf{x}_n, \mathbf{y}_m \rangle$. In what follows, the subscript XY will be dropped from H when no confusion arises. Associate the matrix H with the corresponding (in the standard basis) linear map $H : \mathbb{C}^K \to \mathbb{C}^K$. The generalized coherence [1, 2] of the combined set of unit vectors $C_X \cup C_Y$ is

$$\begin{aligned} \gamma_1^2(C_X \cup C_Y) &= 1 - |G(C_X \cup C_Y)| \\ &= 1 - \begin{vmatrix} I_K & H \\ H^\dagger & I_K \end{vmatrix} \\ &= 1 - \left| I_K - HH^\dagger \right| \end{aligned} \tag{1}$$

This quantity describes a relationship between the subspaces V_X and V_Y that does not depend explicitly on the sets C_X and C_Y, hence it makes sense to denote $\gamma_1^2(V_X, V_Y) = \gamma_1^2(C_X \cup C_Y)$. The cosines of the principal angles $\{\theta_1, \cdots, \theta_K\}$ between V_X and V_Y are the singular values of H_{XY}, so the eigenvalues of $H_{XY} H_{XY}^\dagger$ are $\{\cos^2\theta_1, \cdots, \cos^2\theta_K\}$. Thus,

$$\gamma_1^2(V_X, V_Y) = 1 - \prod_{j=1}^{K} (1 - \cos^2\theta_j).$$

Perfect coherence (i.e., $\gamma_1^2(V_X, V_Y) = 1$) is obtained if any θ_j is zero; i.e., if V_X and V_Y share a one-dimensional subspace.

3. EXTERIOR POWERS OF A VECTOR SPACE

Higher order coherence of vector subspaces is most naturally defined in terms of exterior algebra and particularly, in terms of the exterior powers of a vector space [10]. Let V be a vector space of dimension N with inner product $\langle\ ,\ \rangle$. The p^{th} exterior power of V, denoted by $\bigwedge^p V$ is a subspace of $V^{\otimes p}$ which is spanned by the antisymmetric tensors

$$\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p = \frac{1}{p!}\sum_{\sigma} \mathrm{sgn}(\sigma)\ \mathbf{z}_{\sigma(1)} \otimes \mathbf{z}_{\sigma(2)} \otimes \cdots \otimes \mathbf{z}_{\sigma(p)}.$$

In this expression, $\mathbf{z}_1, \cdots \mathbf{z}_p \in V$ and the sum is over all permutations of $\{1, \cdots, p\}$. The elements of $\bigwedge^p V$ are called p-vectors. Define the set $\mathcal{S}(p, N)$ of all p-element strictly increasing sequences selected from $\{1, \cdots, N\}$ and suppose $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is an orthonormal basis for V. For each element $I = i_1, \cdots, i_p \in \mathcal{S}(p, N)$, define the p-vector

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p}.$$

Then the set $\{\mathbf{e}_I | I \in \mathcal{S}(p, N)\}$ is a basis for $\bigwedge^p V$.

A p-vector that can be written as the exterior product of p elements of V as $\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p$ is called *pure*. The space $\bigwedge^p V$ has an inner product induced by the inner product on V. It is obtained by defining

$$\langle \mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p, \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_p\rangle_p = \left|[\langle \mathbf{z}_i, \mathbf{w}_j\rangle]_{1\le i,j\le p}\right| = \begin{vmatrix} \langle \mathbf{z}_1, \mathbf{w}_1\rangle & \cdots & \langle \mathbf{z}_1, \mathbf{w}_p\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{z}_p, \mathbf{w}_1\rangle & \cdots & \langle \mathbf{z}_p, \mathbf{w}_p\rangle \end{vmatrix}$$

for pure p-vectors and extending to all of $\bigwedge^p V$ by linearity. In terms of this inner product, the basis $\{\mathbf{e}_I | I \in \mathcal{S}(p, N)\}$ is orthonormal.

Some insight into the nature of pure p-vectors is obtained by considering the matrix Z whose i^{th} row consists of components of $\mathbf{z}_i$ in the basis $\{\mathbf{e}_j | j = 1, \cdots, N\}$ for V. The expansion of $\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p$ in the basis $\{\mathbf{e}_I | I \in \mathcal{S}(p, N)\}$ is given by

$$\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p = \sum_{I\in\mathcal{S}(p,N)} Z_I \mathbf{e}_I$$

where the Z_I are the principal minors of Z; i.e.,

$$Z_I = \begin{vmatrix} z_{1,i_1} & z_{1,i_2} & \ldots & z_{1,i_p} \\ z_{2,i_1} & z_{2,i_2} & \ldots & z_{2,i_p} \\ \vdots & \vdots & & \vdots \\ z_{p,i_1} & z_{p,i_2} & \ldots & z_{p,i_p} \end{vmatrix}.$$

In defining p-coherence of subspaces, it will also be necessary to use the concept of exterior powers of linear maps. Suppose that $A : V \to V$ is a linear map. A map $\wedge^p A : \bigwedge^p V \to \bigwedge^p V$ can be defined through its action on pure p-vectors as

$$\wedge^p A\ \ (\mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p) = A\mathbf{z}_1 \wedge \cdots \wedge A\mathbf{z}_p.$$

In the basis $\{\mathbf{e}_I | I \in \mathcal{S}(p, N)\}$, $\wedge^p A$ has matrix elements

$$\langle \mathbf{e}_I, \wedge^p A\ \mathbf{e}_J\rangle_p = \begin{vmatrix} A_{i_1,j_1} & A_{i_1,j_2} & \ldots & A_{i_1,j_p} \\ A_{i_2,j_1} & A_{i_2,j_2} & \ldots & A_{i_2,j_p} \\ \vdots & \vdots & & \vdots \\ A_{i_p,j_1} & A_{i_p,j_2} & \ldots & A_{i_p,j_p} \end{vmatrix}$$

which are the $(I, J)^{\text{th}}$ minors of the matrix for A. For two linear maps $A, B : V \to V$, the exterior power has the property that

$$\wedge^p(AB) = (\wedge^p A)(\wedge^p B).$$

An important property of pure p-vectors is their relationship to p-dimensional subspaces of V [10, 11]. Let $\{\mathbf{z}_1, \ldots, \mathbf{z}_p\}$ be a linearly independent set of p-vectors in V. These vectors form a basis for a subspace $V_Z \subset V$. If B is any non-singular matrix, then the vectors

$$\mathbf{z}'_k = \sum_{j=1}^{p} B_{ij}\mathbf{z}_j \tag{2}$$

are also a basis for V_Z. Associated with $\{\mathbf{z}_1, \ldots, \mathbf{z}_p\}$ is the p-vector $\boldsymbol{\zeta} = \mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p$. Under the change of basis (2),

$$\boldsymbol{\zeta}' = \mathbf{z}'_1 \wedge \cdots \wedge \mathbf{z}'_p = |B|\boldsymbol{\zeta}.$$

Thus the subspace V_Z is associated with a ray in $\bigwedge^p V$ in the direction $\boldsymbol{\zeta}$; i.e., $\{\alpha\boldsymbol{\zeta} \mid \alpha \in \mathbb{C}\}$. In general there is a bijection between p-dimensional subspaces of V and rays in $\bigwedge^p V$ in the direction of pure p-vectors. Given two pure p-vectors $\boldsymbol{\zeta} = \mathbf{z}_1 \wedge \cdots \wedge \mathbf{z}_p$ and $\boldsymbol{\omega} = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_p$ then in terms of the inner product $\langle\ ,\ \rangle_p$, the angle between the subspaces associated with these p-vectors is

$$\cos\theta = \frac{|\langle \boldsymbol{\zeta}, \boldsymbol{\omega}\rangle_p|}{\sqrt{\langle \boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle_p}\sqrt{\langle \boldsymbol{\omega}, \boldsymbol{\omega}\rangle_p}}.$$

It is obviously invariant under change of basis for either of the subspaces. This angle between p-dimensional subspaces is related to the principal angles $\theta_1, \cdots, \theta_p$ between the subspaces by

$$\cos\theta = \prod_{j=1}^{p} \cos\theta_j.$$

This is proved in [12] and is a special case of Theorem 1 below.

4. THE p-COHERENCE OF SUBSPACES

Consider again the sets C_X and C_Y from Section 2. Define a set $C_X^p \subset \bigwedge^p V$ by

$$C_X^p = \{\mathbf{x}_I = \mathbf{x}_{i_1} \wedge \cdots \wedge \mathbf{x}_{i_p} | I \in \mathcal{S}(p, K)\}.$$

The elements of C_X^p span the subspace $\bigwedge^p V_X \subset \bigwedge^p V$. Similarly define C_Y^p and $\bigwedge^p V_Y$ from C_Y. The cross gram matrix between C_X^p and C_Y^p is given by

$$[H_{XY}^p]_{I,J} = \langle \mathbf{x}_I, \mathbf{y}_J\rangle_p, \quad \text{for } I, J \in \mathcal{S}(p, K).$$

Note that H_{XY} is a linear map $\mathbb{C}^K \to \mathbb{C}^K$ and that H_{XY}^P is a linear map $\bigwedge^p \mathbb{C}^K \to \bigwedge^p \mathbb{C}^K$. In fact, is can be seen that

$$H_{XY}^p = \wedge^p H_{XY}.$$

With this notation, it is now possible to define p^{th} order coherence between K-dimensional subspaces V_X and V_Y of an N-dimensional complex vector space V, as follows.

Definition 1. *Let V be an N-dimensional vector space over $\mathbb{C}$. The p^{th} order coherence (p-*coherence*) between two K-dimensional ($K < N$) subspaces V_X and V_Y of V is*

$$\gamma_p^2(V_X, V_Y) = \gamma_1^2\big(\textstyle\bigwedge^p V_X, \bigwedge^p V_Y\big).$$

Applying (1) to this definition gives

$$\begin{aligned}\gamma_p^2(V_X.V_Y) &= 1 - |G(C_X^p \cup C_Y^p)| \\ &= 1 - \begin{vmatrix} I_{\binom{K}{p}} & \wedge^p H \\ (\wedge^p H)^\dagger & I_{\binom{K}{p}} \end{vmatrix} \\ &= 1 - \left| I_{\binom{K}{p}} - \wedge^p \left(H H^\dagger \right) \right|\end{aligned}$$

This expression for $\gamma_p^2(V_X, V_Y)$ leads to the following result:

Theorem 1. *The p-coherence of two K-dimensional subspaces V_X and V_Y of an N-dimensional ($N > K$) complex vector space V is given by*

$$\gamma_p^2(V_X, V_Y) = 1 - \prod_{I \in \mathcal{S}(p,K)} (1 - \prod_{i \in I} \cos^2 \theta_i)$$

where the θ_i, $i = 1, \cdots, K$, are the principal angles between V_X and V_Y.

Corollary 1. *Perfect p-coherence, $\gamma_p^2(V_X, V_Y) = 1$, occurs only if V_X and V_Y share a p-dimensional subspace.*

Proof. Theorem 1 follows from the fact that, if a linear map $A : \mathbb{C}^K \to \mathbb{C}^K$ has (right) eigenvectors $\{\mathbf{w}_1, \cdots, \mathbf{w}_K\}$ with corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_K\}$, then the map $\wedge^p A : \bigwedge^p \mathbb{C}^K \to \bigwedge^p \mathbb{C}^K$ has (right) eigenvectors

$$\{\mathbf{w}_I = \mathbf{w}_{i_1} \wedge \cdots \wedge \mathbf{w}_{i_p} | I \in \mathcal{S}(p, K)\}$$

with corresponding eigenvalues

$$\{\lambda_I = \lambda_{i_1} \cdots \lambda_{i_p} | I \in \mathcal{S}(p, K)\}. \tag{3}$$

□

Incidentally, the form of the eigenvalues of $\wedge^p A$ in (3) implies the useful relation

$$|\wedge^p A| = |A|^{\binom{K-1}{p-1}} \tag{4}$$

The p-coherence has so far been defined for a pair of subspaces. The p-coherence can be generalized in the following way. Suppose there are L sets $C_{X_1}, C_{X_2}, \cdots, C_{X_L}$ each consisting of K vectors in V. Associated with each C_{X_j} there a subspace V_{X_j} spanned by the vectors in C_{X_j}. The p-coherence for $V_{X_1}, V_{X_2}, \cdots, V_{X_L}$ is defined as

$$\gamma_p^2(V_{X_1}, \cdots, V_{X_L}) = 1 - |G(C_{X_1}^p \cup C_{X_2}^p \cup \cdots \cup C_{X_L}^p)|$$

This multi-subspace p-coherence will be considered in detail in future work. The remainder of this paper considers only coherence for pairs of vector spaces.

5. RELATIONSHIP TO CANONICAL CORRELATIONS

Let $\mathcal{H}$ be the Hilbert spaces of complex-valued random variables on some probability space. The inner product between two random variables X and $Y \in \mathcal{H}$ is

$$\langle X, Y \rangle = \mathsf{E}\{X^* Y\} \tag{5}$$

Consider two sets of K zero-mean random variables $C_X = \{X_1, \cdots, X_K\}$ and $C_Y = \{Y_1, \cdots, Y_K\}$. Associated with the sets C_X and C_Y are the subspace V_X and $V_Y \subset \mathcal{H}$ consisting to the linear spans of the set of random variables. The canonical correlation coefficients [13] between C_X and C_Y, denoted by ρ_j, $j = 1, \cdots, K$, are the singular values of $R_{XX}^{-1/2} R_{XY} R_{YY}^{-1/2}$, where R_{XX} and R_{YY} are the covariance matrices of X and Y, respectively, and R_{XY} denotes their cross-covariance matrix. The canonical correlation coefficients are actually the cosines of the principal angles between the subspaces V_X and V_Y with respect ti the inner product (5). The coherence between two subspaces V_X and $V_Y \in \mathcal{H}$ is defined by

$$\gamma_1^2(V_X, V_Y) = 1 - \frac{\begin{vmatrix} R_{XX} & R_{XY} \\ R_{XY}^\dagger & R_{YY} \end{vmatrix}}{|R_{XX}||R_{YY}|}$$

and the p-coherence between V_X and V_Y is defined as

$$\begin{aligned}\gamma_p^2(V_X, V_Y) &= \gamma_1\big(\bigwedge^p V_X, \bigwedge^p V_Y\big)^2 \\ &= 1 - \frac{\begin{vmatrix} \wedge^p R_{XX} & \wedge^p R_{XY} \\ \wedge^p R_{XY}^\dagger & \wedge^p R_{YY} \end{vmatrix}}{|\wedge^p R_{XX}||\wedge^p R_{YY}|}\end{aligned}$$

In a similar way to the proof of Theorem 1 it can be shown that

$$\begin{aligned}&\gamma_p^2(V_X, V_Y) \\ &= 1 - \left| I_{\binom{K}{p}} - \wedge^p \left(R_{XX}^{-1/2} R_{XY} R_{YY}^{-1} R_{XY}^\dagger R_{XX}^{-1/2} \right) \right| \\ &= 1 - \prod_{I \in \mathcal{S}(p,K)} (1 - \prod_{i \in I} \rho_i).\end{aligned}$$

6. DISTRIBUTION OF SOME STATISTICS UNDER THE NULL HYPOTHESIS

In using γ_p^2 to detect the presence of higher-order correlation between two data sets C_X and C_Y, it is advantageous to know the distribution of the p-coherence statistic under suitable null hypotheses. This section provides preliminary results in this direction.

The vectors in each of the sets C_X and C_Y are assume to have been orthonormalized and so comprise a set of K orthonormal vectors in V. Such a set is known as a K-*frame*, and the collection of all K-frames in V defines a smooth manifold $\mathcal{V}_K(V)$ of complex dimension $2\dim(V)K - K^2$ called a Stiefel manifold [14]. A suitable null hypothesis is that each of the of the sets is a uniformly distributed K-frame, according to the invariant measure on $\mathcal{V}_K(V)$, and that the two K-frames are independently distributed. This will be the case if each of the vectors in C_X and C_Y are independently drawn from the complex gaussian distribution $\mathcal{CN}(0, I_N)$ and the vectors in each of C_X and C_Y separately orthonormalized.

Under this null hypothesis, the quantity

$$Q = H_{XY} H_{XY}^\dagger \sim \mathcal{B}_K(K, N-K)$$

is matrix beta distributed. Note that $(I - Q) \sim \mathcal{B}_K(N-K, K)$. Factorize Q as $Q = T^\dagger T$, where T is upper triangular with positive diagonal entries. Adapting Theorem 3.3.3 in [15] (See [16]) to the complex case implies that the squares of the diagonal entries $\tau_j = t_{jj}^2$, $j = 1, \cdots, K$, of T are independently beta distributed with $\tau_j \sim \mathcal{B}(N-K-j+1, K)$. Thus

$$|H_{XY} H_{XY}^\dagger| \sim \prod_{j=1}^K \tau_j \tag{6}$$

where $\tau_j \sim \mathcal{B}(N-K-j+1, K)$, for $j = 1, \cdots, K$. In a similar way

$$\gamma_1^2 = 1 - |I - Q| \sim 1 - \prod_{j=1}^{K} \beta_j$$

where $\beta_j \sim \mathcal{B}(K-j+1, N-K)$, for $j = 1, \cdots, K$, independently. This result also follows in a direct way from [17].

A useful result in analyzing p-coherence statistics is the following.

Theorem 2. *Let C_X and C_Y consist of independent uniformly distributed K-frames in $\mathcal{V}_K(V)$, and let H_{XY} be the cross gram matrix of C_X and C_Y. The distribution of the determinant $|\bigwedge^p(H_{XY}H_{XY}^\dagger)|$ is given by*

$$|\wedge^p (H_{XY}H_{XY}^\dagger)| \sim \prod_{j=1}^{K} \tau_j^{\binom{K-1}{p-1}}$$

where the β_j are independently beta distributed as $\beta_j \sim \mathcal{B}(N-K-j+1, K)$, for $j = 1, \cdots, K$.

Proof. Using (4),

$$|\wedge^p (H_{XY}H_{XY}^\dagger)| = |H_{XY}H_{XY}^\dagger|^{\binom{K-1}{p-1}} \sim \prod_{j=1}^{K} \tau_j^{\binom{K-1}{p-1}}$$

where $\tau_j \sim \mathcal{B}(N-K-j+1, K)$, for $j = 1, \cdots, K$ according to (6). □

Theorem 2 implies that

$$\gamma_K^2 \sim 1 - \prod_{j=1}^{K} \tau_j$$

where $\tau_j \sim \mathcal{B}(N-K-j+1, K)$, for $j = 1, \cdots, K$. The distributions of γ_p^2 for $1 < p < K$, under the null hypothesis remain to be determined.

7. NUMERICAL RESULTS

This section evaluates the performance of the detection statistics given in γ_p^2 by means of simulations.

Consider a scenario in which two sets of $K = 4$ channels are monitored and $N = 128$ or $N = 256$ samples per channel are collected. The distribution of γ_p^2 under the null hypothesis was investigated by drawing each N-sample vector independently from a $\mathcal{CN}(0, I_N)$ distribution. Over 5×10^6 realizations, the values of γ_p^2 for $p = 1, 2, 3,$ and 4 were collected. The resulting measured cumulative distributions for $N = 128$ are displayed in Figure 1.

Three scenarios where used in testing performance of p-coherence as a detection statistic. These were

- Rank-0 For each realization, each N-sample vector is independently realized from a $\mathcal{CN}(0, I_N)$ distribution
- Rank-1 A single non-zero (0.2) canonical correlation is introduced between the two sets of channels
- Rank-2 Two non-zero (0.2, 0.3) canonical correlations are introduced between the two sets of channels

The performance of the detector was evaluated for two cases: rank-2 against rank-0 and rank-2 against rank-1. Figure 2 shows the corresponding receiver operating characteristic (ROC) curves for probability of correct detection of rank-2 against probability of false alarm when testing rank-2 against rank-0, and rank-2 tested against rank-1.

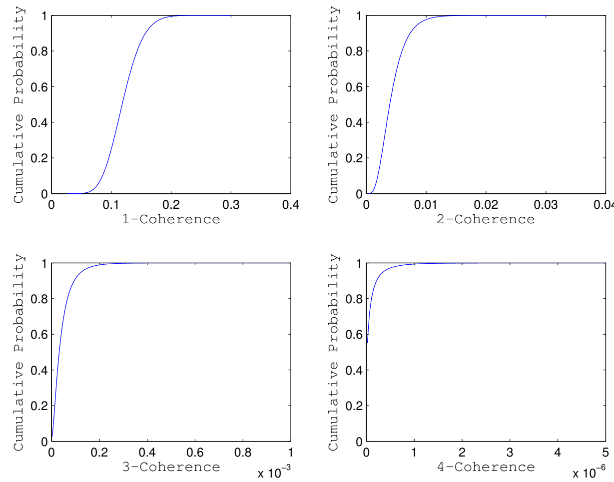

Fig. 1. Cumulative distribution for $1, 2, 3$ and 4-coherence under the null hypothesis.

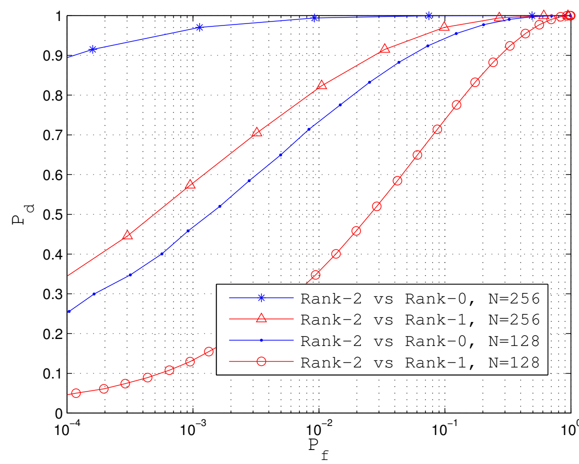

Fig. 2. ROC for two sets of 4 receivers, $N = 128$ and 256, $p = 1$ with $\cos\theta_1 = 0.2$ and $p = 2$ with $\cos\theta_1 = 0.2$ and $\cos\theta_2 = 0.3$.

8. CONCLUSION

This paper has introduced p-coherence and examined some of the properties and relationships to standard concepts in statistical signal processing. Its use as a multiple-channel detection statistic has been proposed, some preliminary results regarding its distribution under a suitable null hypothesis has been given, and simulations support its potential applicability in certain scenarios. But much more complete examination of p-coherence as a detection statistic is warranted.

9. ACKNOWLEDGMENT

This work was supported in part by the U.S. Air Force Office of Scientific Research under Grant No. FA9550-12-1-0225.

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