# Gauge-invariant Registration in Networks

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***Abstract*—Integration and exploitation of information collection across a distributed network of assets usually requires the establishment and maintenance of registration of coordinates across the nodes of the network. Here "registration" covers a range of possibilities, including clock synchronization and registration of frames of reference. The registration problem is posed in terms of network represented by a graph $\Gamma$ with vertices corresponding to the sensors. Attached to the edges of the graph are noisy measurements of the "difference" between the two coordinate systems. This "difference" is expressed in terms of a member of a Lie group of coordinate transformations. Effectively, the registration problem is specified in terms of a connection on the edges, and becomes one of estimating a gauge transformation to align the coordinate systems across the network. The key descriptor of the difficulty of the estimation problem, the Fisher information, can be simply expressed in terms of the geometry of the situation and provides a link between the homological chains and cochains for the graph.**

## I. Introduction

Registration of data across a network is a ubiquitous problem in distributed sensing. Over more than three decades, much effort has been expended on development of algorithms to provide time synchronization across a distributed network; e.g., [1], [2], [3], [4], [5]. Synchronization of this kind is important for distributed parallel processing as well as data fusion across a sensor network. It is typically the case that the network is not complete; i.e., each node does not communicate with every other node. A large fraction the approaches described in the literature produce algorithms to minimize an error or objective function based on least squares, often within power or other resource constraints. Leaving aside the latter issue, the problem in this setting is to assign an adjustment to the datum at each node based on knowledge of the differences, generally noisy, between the values at some pairs of nodes in the network. Even if difference measurements are available for every pair of nodes in the network, the presence of noise still raises consistency considerations; e.g., the true offsets must sum to zero around any closed cycle.

The domain of practical network synchronization problems is by no means limited to situations in which the natural measurement space is the real line. Individual nodes may possess multiple data to be registered across the network, and the noise affecting such vector data may be correlated across its components. Research in this arena has led to sophisticated distributed algorithms for estimation problems that apply beyond the realm of network alignment [6], [7], [8], [9], but little attention has been given to situations in which the natural measurement space is a Lie group [10], [11] rather than a linear space. In phase synchronization, for example, typical data could be measurements of the phase differences between local oscillators at the nodes. In this setting, the natural measurement space is the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ rather then the real line $\mathbb{R}$. If several local oscillators are involved, measurements might lie on the torus $\mathbb{T}^n$. Another important practical example where the measurement space is a nonlinear multi-dimensional manifold is registration of local coordinate systems, for which the natural setting in the special orthogonal group $SO(3)$. In the context of clock synchronization, if both offset and clock speed are adjustable locally then the offsets are elements of the affine group $\mathbb{A}$. These examples illustrate that practical problems can entail data on Lie groups that are compact (e.g., $\mathbb{T}$ or $SO(3)$), non-compact ($\mathbb{R}^n$), abelian ($\mathbb{R}^n$, $\mathbb{T}$), or non-abelian ($SO(3)$ or $\mathbb{A}$).

It is common to represent networks in terms of graphs. In this setting, the network nodes that provide data to be registered or synchronized are represented by vertices labeled with their associated parameters, such as local clock time or local coordinate system. Each pair of vertices corresponding to a pair of nodes that are in direct communication are joined by an edge. Information is

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shared between vertices along such edges, each of which is labeled by a noisy measurement of the coordinate transformation needed to take a value from the vertex at the start of the edge into the coordinate system of the vertex at the end. A goal of this paper is to show how such sensor network registration problems are naturally formulated in terms of what is called *gauge theory* in physics or the *theory of connections on principal bundles* in mathematics [12], [13], and how gauge invariance of statistical models can be exploited to better understand the statistical limits of network registration.

The work of Amari and others [14] on the use of methods of Riemannian geometry to analyze statistical estimation problems is of increasing interest to researchers in signal processing. This methodology, known as *information geometry*, provides a rigorous geometric framework for measuring the power of data to discriminate values of parameters. These ideas date back to Rao [15], who showed that the Fisher information of a likelihood used in an estimation problem can be seen as a Riemannian metric on the parameter manifold. This paper elucidates the implications of gauge invariance in the information geometry of statistical estimation problem associated with sensor network registration.

## II. Network Registration and Gauge Transformations

The basic model represents a sensor network by a graph $\Gamma$ with vertex set $V(\Gamma)$ indexing the sensors and edge set $E(\Gamma)$, as defined below. Each sensor makes observations that are represented in some local coordinate system. There exists a group $G$ of transformations that convert between alternate coordinate systems. This group is referred to as the *gauge group*. A priori, there is no preferred choice of coordinate system or even a preferred reference coordinate system. The act of choosing a reference coordinate system at each node in the network is called choosing a *gauge* (or *trivialization* in mathematics). Having chosen a gauge, the state of the sensor network, in terms of the coordinate system chosen, can be represented as a point in the Cartesian product $V(\Gamma) \times G$; i.e., the state of each node is represented by the group transformation that takes its reference coordinate system to the one in use. It is possible to mathematically describe the state of the network without a choosing a specific gauge in terms of a principal $G$-bundle with discrete base space $V(\Gamma)$ [12]. The state of the network is a section of this bundle.

Each pair of vertices corresponding to a pair of nodes that are in direct communication are joined by an edge in $\Gamma$. Information is shared between vertices along such edges, each of which, in the absence of noise, is labeled by the map taking the state of the vertex at the start of the edge to the state of the vertex at the end. The collection of such maps over $E(\Gamma)$ is called a *connection*.

In the network registration/alignment problem it is assumed that there exist global common, or aligned, coordinate systems; i.e., that there are choices of gauge such that, when each of the nodes is using the reference coordinate system, the value of the connection on all edges is the identity transformation in $G$. Such a network will be called *alignable*. When the network is not aligned, its state in this gauge is an element of $V(\Gamma) \times G$, where each node is associated with the transformation that takes the common reference coordinate system to the coordinate system of the node.

A *gauge transformation* is separate change of coordinate system at each node in the network. In a particular gauge, such a transformation is an element of $V(\Gamma) \times G$. In the case of a noise free-observation of the connection, network registration for an alignable network corresponds to finding a gauge transformation that takes the observed connection to the identity connection.

In problems considered in this paper, the observation of the connection is noisy. So the network registration problem becomes one of estimating a gauge transformation that aligns the network. That is, finding a gauge transformation for which the resulting connection is a close as possible, by some statistical measure, to the identity connection. It will be shown in Section IV that, for a natural noise model, this problem becomes a gauge-invariant estimation problem. Before this, some general results for invariant estimation on Lie groups will be derived.

## III. Information Geometry on Lie Groups

In this section we develop some general results for invariant estimation on Lie groups which we will need subsequently. Suppose that $X$ is a random variable with sample space $\mathcal{X}$ and that $G$ is a Lie group such that

1) There is a left action of $G$ on $\mathcal{X}$
2) There is a left $G$-invariant measure $\mu$ on $\mathcal{X}$

We are given a parameterized set of possible statistical models for an observation of $x \in \mathcal{X}$, $\mathcal{S} = \{f(x|g) \mid g \in G\}$, where the $f(x|g)$ are probability densities relative to $\mu$. These densities are assumed to be smooth as functions on $G$. The statistical problem is estimation of $g \in G$ given a realization of the data $x \in \mathcal{X}$.

*Definition 1:* A family of densities $\mathcal{S}$ is said to be invariant under a group $G$, if for every $g, h \in G$, $Y = hX$ has density $f(y|hg)$.

We will assume in what follows that the family $\mathcal{S}$ is invariant under $G$ and consider the problem of estimating the value of the parameter $g \in G$ given a observation of $X$.

The Kullback-Leibler divergence $D : G \times G \to \mathbb{R}^+$ measures how good our measurement is expected to be at distinguishing two values $g$ and $g'$ of the parameter, given that the value is really $g$. For this estimation problem

$$D(g\|g') = \int_{\mathcal{X}} f(x|g) \log \frac{f(x|g)}{f(x|g')} d\mu(x)$$

where we make the additional assumption on the densities that this integral exists, is finite for all $g, g' \in G$, and is $C^2$ in each variable. This is a mild extra assumption that is satisfied for statistical distributions of practical interest. Invariance under $G$ implies that

$$D(hg\|hg') = D(g\|g') \qquad (1)$$

for all $h \in G$ and, in particular, that

$$D(g\|g') = D(\mathrm{id}\|g^{-1}g') = D(g'^{-1}g\|\mathrm{id})$$

where $\mathrm{id}$ denotes the identity element of $G$.

The Fisher information metric can be defined as

$$\begin{aligned} F_g &= \nabla^2 D(\cdot\|g')\big|_{g=g'} = \nabla^2 D(g'\|\cdot)\big|_{g=g'} \\ &= \int_{\mathcal{X}} (d\log f)_g \otimes (d\log f)_g \; f(x|g) d\mu(x). \end{aligned} \qquad (2)$$

where $\nabla$ is any covariant derivative on $G$ (all give the same result). This quantifies how good a measurement at distinguishing "infinitesimal" changes in the parameter $g$.

For the invariant estimation problem the Fisher metric on $G$ is left invariant; i.e.,

$${L_h}^* F = F$$

where $L_h$ is the left action of $h \in G$ on $G$. Here ${L_h}^*$ denotes the pullback corresponding to $L_h$.

A special case arises when $f$ is a fixed distribution on $\mathcal{X}$ and the family of distributions is defined by $f(x|g) = f(g^{-1}x)$. It is easily checked that this is an invariant family. The corresponding Fisher Information is called the *distributional Fisher Information* of $f$, and quantifies the ability of a measurement to distinguish "infinitesimal" shifts in the distribution by the group action of $G$ on $\mathcal{X}$. This concept has been widely studied, albeit typically in less generality than described here; see, for instance, [16].

The Cartan-Maurer one-form $\sigma$ is a $\mathfrak{g}$-valued one form on $G$, where $\mathfrak{g}$ denotes the Lie algebra of $G$. Its value at $g \in G$, $\sigma_g : T_g G \to T_{\mathrm{id}} G = \mathfrak{g}$ associates with any $u \in T_g G$, the element of $\mathfrak{g}$ corresponding to the unique left invariant vector field on $G$ whose value at $g \in G$ is precisely the given tangent vector $u$. For any pair of vector fields $U$ and $V$ on $G$, $\sigma$ satisfies the Cartan-Maurer equation

$$d\sigma(U, V) + [\sigma(U), \sigma(V)] = 0.$$

A consequence of the invariance of the Fisher information metric is that, for vector fields $U$ and $V$ on $G$,

$$F(U, V) = F_{\mathrm{id}}(\sigma(U), \sigma(V)).$$

Thus, we need only compute the Fisher information at the identity element of $G$. Choose a basis $(E_a)_{a=1}^{\dim G}$ for $\mathfrak{g}$ regarded as the left invariant vector fields on $G$. Then the one-form $\sigma$ can be written as

$$\sigma = \sum_{a=1}^{\dim G} \sigma_a E_a$$

where the coefficients $\sigma_a \in T^*G$ are regular one-forms on $G$. The Fisher metric takes the form

$$F = \sum_{a,b=1}^{\dim G} \mathcal{F}_{ab} \; \sigma_a \otimes \sigma_b$$

where the coefficients $\mathcal{F}_{ab} = F_{\mathrm{id}}(E_a, E_b)$ are constant.

## IV. Alignment for Unimodular Lie Groups as Gauge Theory on Graphs

### A. Gauge Theory on Graphs

We now come back to the network alignment problem. We will assume that the group $G$ is unimodular: that is, its left and right Haar measures are identical. This property holds for all compact groups and all abelian groups, as well as many others. The network is represented by a graph $\Gamma$ with edge set $E(\Gamma)$ and vertex set $V(\Gamma)$, as described above. In this section we give a summary of Baez [17], who formulated gauge theory on graphs in relation to spin networks in physics. In subsequent sections we show how this formalism can be used to develop a theory of gauge invariant estimation for sensor network alignment.

Define $P$ as a principal $G$-bundle over $V(\Gamma)$, regarding $V(\Gamma)$ as a topological space with the discrete topology. This is a fiber bundle $\pi : P \to G$ along with a continuous right action of $G$ on $P$ which preserves the fibers of $P$ and acts freely and transitively on them. For $v \in V(\Gamma)$ denote by $P_v$ the fiber over $v$, that is, the set of elements of $p \in P$ satisfying $\pi(p) = v$. Since $V(\Gamma)$ is discrete $P$ is globally trivializable, that is, one can chose a global gauge or fixed reference element for each fiber so that $P = V(\Gamma) \times G$. Thus, trivialization allows identification of the fibers $P_v$ with copies of $G$. However, for the moment we leave the gauge unspecified.

For an edge $e \in E(\Gamma)$ define $\mathcal{A}_e$ the space of smooth maps (coordinate transforms) $a : P_{s(e)} \to P_{t(e)}$, compatible with the right action of $G$ on $P$

$$a(xg) = a(x)g$$

Here $s : E(\Gamma) \to V(\Gamma)$ and $t : E(\Gamma) \to V(\Gamma)$ give the *source* and *target* nodes of edge $e$. The space of *connections* is

$$\mathcal{A} = \prod_{e \in E} \mathcal{A}_e$$

Trivialisation allows identification of each $\mathcal{A}_e$ with a copy of $G$ having elements as maps from $P_{s(e)} \cong G$ to $P_{t(e)} \cong G$. This allows the identification of $\mathcal{A}$ with $G^E$. Thus, the space of connections $\mathcal{A}$ has the the structure of a smooth manifold. Furthermore, it inherits an invariant measure: the product of $|E(\Gamma)|$ copies of the Haar measure on $G$. This manifold structure and measure are independent of trivialization.

Define the group of gauge transformations on $P$ by

$$\mathcal{T} = \prod_{v \in V} \left(P_v \times_{\text{Ad}} G\right).$$

where $\times_{\text{Ad}}$ means that we take $(ph, h^{-1}uh) \sim (p, u)$, for all $h \in G$, $p \in P_v$ and $u \in G$. Trivialization of $P$ lets us identify $\mathcal{T}$ with $G^V$. To see this, we note that by trivialization $g_v \in P_v \times_{\text{Ad}} G$ can be written as

$$((v, g), u) \sim ((v, \text{id}), g^{-1}ug) \tag{3}$$

for some $g$ and $u$ in $G$.

The group $\mathcal{T}$ acts on the space of connections as follows. Write $g_v$ for value of $g \in \mathcal{T}$ at $v$. Then, regarding $g_v$ as a map from $P_v$ to itself, $\mathcal{T}$ acts on $\mathcal{A}$ by

$$(g\mathcal{A})_e = g_{t(e)} \mathcal{A}_e g_{s(e)}^{-1}.$$

### B. Gauge Invariant Estimation

We now show how gauge theory on graphs can be used to formulate the network registration problem as a gauge invariant estimation problem and discuss some consequences of this.

In terms of the alignment problem the state of the sensor network is a section of $P$. This state is never directly observed.

What is observed, ideally, is the connection $r \in \mathcal{A}$. This is assumed to be achieved by a pair-wise comparison of the nodes in the network by some means such as comparing observations of some specified collection of objects. However, in practical situations, observations are always contaminated by noise. Each pair of nodes which have been compared are taken to be joined by an edge labeled by the observed connection value. A network is *alignable* if the noise free connection is flat. That is, if there is a gauge transformation that takes noise free connection to the identity connection.

We take the noise on the measurements to be a random variable taking values in $\varepsilon \in \mathcal{A}$ with probability density $p(\varepsilon)$ with respect to the above-mentioned normalized invariant measure on $\mathcal{A}$. The measurement on the edge $e$ takes the form

$$r_e = g_{t(e)} \varepsilon_e g_{s(e)}^{-1}$$

Consequently, the probability density for the measured connection is

$$f(r|g) = p(g^{-1}r) \tag{4}$$

Our problem now becomes one of estimating a gauge transformation $g \in \mathcal{T}$ given a noisy measurement of the connection $r \in \mathcal{A}$. We need to be a careful here with regard to parameterization and observability. For example, if the noise on the connection is independent between edges then

$$f(r|g) = \prod_{e \in E(\Gamma)} p_e(g_{s(e)} r_e g_{t(e)}^{-1}) \tag{5}$$

where the $p_e$ are the probability densities for each edge. If $G$ is abelian or if the densities $p_e$ are class functions of the $G$ then certain subgroups of $\mathcal{T}$ do not change $f(r|g)$. In this case, $g$ can only be estimated up to a subgroup $\mathcal{T}_0$ and the parameter space is $\mathcal{T}/\mathcal{T}_0$. Here we will assume that the $f(r|g)$ does not have such symmetries and take the parameter space to be $\mathcal{T}$, however, we shall come back to this point when we consider abelian gauge groups $G$ in a later section.

We have a gauge invariant estimation problem; i.e.,

1) A left action of the group $\mathcal{T}$ on $\mathcal{A}$
2) A left $\mathcal{T}$-invariant measure on $\mathcal{A}$
3) The probability density $f(r|g)$ in (4) is $\mathcal{T}$-invariant.

By the results of Section III the Fisher information metric on $\mathcal{T}$ is invariant under the left action of $\mathcal{T}$ on itself

$$L_h^* F = F \tag{6}$$

for all $h \in \mathcal{T}$. Thus for vector fields $U$ and $V$ on $\mathcal{T}$,

$$F(U,V) = F_{\mathrm{id}}(\sigma(U), \sigma(V))$$

where $\sigma$ is the Cartan-Maurer one-form on $\mathcal{T}$ and

$$F_{\mathrm{id}} = \int_{\mathcal{A}} d\log f(r|.)|_{\mathrm{id}} \otimes d\log f(r|.)|_{\mathrm{id}}\ f(x|\mathrm{id})\, d\mu(r)$$

and $\mu$ denotes the $\mathcal{T}$-invariant measure on $\mathcal{A}$.

If the noise connection is independently distributed on edges, so that $f(r|g)$ takes the form (5) then $F$ can be further decomposed. Define the incidence projection $\mathcal{D}_e : \mathcal{T} \to \mathcal{T}_e$ by

$$\mathcal{D}_e(g) = (g_{s(e)}, g_{t(e)}). \tag{7}$$

Then the Fisher metric at $\mathrm{id} \in \mathcal{T}$ is

$$F_{\mathrm{id}} = \sum_{e\in E} \mathcal{D}_e^* F_{\mathrm{id}}^e.$$

So, for vector fields $U, V \in \mathcal{T}$,

$$F(U,V) = \left(\sum_{e\in E} \mathcal{D}_e^* F_{\mathrm{id}}^e\right)\ (\sigma(U), \sigma(V)) \tag{8}$$

where $F_{\mathrm{id}}^e \in T_{\mathrm{id}}^*\mathcal{T}_e \otimes T_{\mathrm{id}}^*\mathcal{T}_e$ is a Fisher information associated with edge $e$ given by

$$F_{\mathrm{id}}^e = \int_{\mathcal{A}_e} d\log p_e(g_2 r_e g_1^{-1})|_{\mathrm{id}} \otimes d\log p_e(g_2 r_e g_1^{-1})|_{\mathrm{id}} \times p_e(g_2 r_e g_1^{-1}) d\mu(r_e).$$

### *C. Abelian Lie Groups*

As remarked above, the case of an abelian gauge group $G$ is a little different. We will assume that the noise on the connection is independent on each edge. The probability density for the observed noisy connection is

$$f(r|g) = \prod_{e\in E(\Gamma)} p_e(r_e g_{s(e)} g_{t(e)}^{-1}).$$

The parameter space in this case is

$$\tilde{\mathcal{T}} = \mathcal{T}/\mathcal{T}_0$$

where $\mathcal{T}_0 \subset \mathcal{T}$ is the subgroup of uniform gauge transformations; i.e., gauge transformations in which the same element of $G$ is applied to each node in $\Gamma$. Since $G$ is abelian, $\mathcal{T}_0$ is a normal subgroup of $\mathcal{T}$. Thus, the results of the previous section apply with the group $\tilde{\mathcal{T}}$ replacing $\mathcal{T}$. In fact, letting $\iota : \tilde{\mathcal{T}} \to \mathcal{T}$ be the inclusion corresponding to our choice of parametrisation of $\tilde{\mathcal{T}}$, the Fisher metric at $\mathrm{id} \in \tilde{\mathcal{T}}$ is

$$F_{\mathrm{id}} = \sum_{e\in E} \iota^* \mathcal{D}_e^* F_{\mathrm{id}}^e.$$

So, for vector fields $U, V \in \tilde{\mathcal{T}}$,

$$F(U,V) = \left(\sum_{e\in E} \iota^* \mathcal{D}_e^* F_{\mathrm{id}}^e\right)\ (\sigma(U), \sigma(V)) \tag{9}$$

where $\sigma$ is the Cartan-Maurer one-form on $\tilde{\mathcal{T}}$. However, in the case of an abelian gauge group there is an illuminating relationship between the Fisher information at the identity in $\tilde{\mathcal{T}}$ and the topology of the graph which we now discuss.

Define 0-chains as the space $C_0(\Gamma, \mathfrak{g}^*) = \mathfrak{g}^* \otimes C_0(\Gamma, \mathbb{R})$, where $\mathfrak{g}$ is the space of invariant one-forms on $G$, which is the dual of the Lie algebra $\mathfrak{g}$. A typical element of $C_0(\Gamma, \mathfrak{g})$ has the form

$$\boldsymbol{x} = \sum_{i=1}^{|V(\Gamma)|} x^i \otimes \boldsymbol{v}_i \tag{10}$$

for $x^i \in \mathfrak{g}^*$ for $i = 1, \cdots, |V(\Gamma)|$, where $\boldsymbol{v}_i$ denotes the basis vector corresponding to the vertex $v_i$. $C_0(\Gamma, \mathfrak{g}^*)$ can be identified with the Lie algebra of the group of gauge transformations $\mathcal{T}$. Similarly, $C_1(\Gamma, \mathfrak{g}^*)$ is the real vector space of 1-chains

$$\boldsymbol{z} = \sum_{i=1}^{|E(\Gamma)|} z^i \otimes \boldsymbol{e}_i \tag{11}$$

for $z^i \in \mathfrak{g}^*$ for $i = 1, \cdots, |E(\Gamma)|$, where $\boldsymbol{e}_i$ denotes the basis vector corresponding to the edge $e_i$.

The *1-boundary map* $\partial_1 : C_1(\Gamma, \mathfrak{g}^*) \to C_0(\Gamma, \mathfrak{g}^*)$ is defined by its action on basis vectors: if the edge $e = (u, v)$ then

$$\partial_1 x \otimes \boldsymbol{e} = x \otimes (\boldsymbol{v} - \boldsymbol{u}) \tag{12}$$

for all $x \in \mathfrak{g}^*$, and again can be extended to the whole of $C_1(\Gamma, \mathfrak{g}^*)$ by linearity. Associated with $\Gamma$ is a *chain complex*; that is a sequence of linear maps $\partial_i$ $(i = 0, 1, 2)$ for which $\partial_i \partial_{i+1} = 0$ for $i = 0, 1$, written as:

$$\{0\} \xleftarrow{\partial_0} C_0(\Gamma, \mathfrak{g}^*) \xleftarrow{\partial_1} C_1(\Gamma, \mathfrak{g}^*) \xleftarrow{\partial_2} \{0\} \tag{13}$$

The 0-boundary map $\partial_0$ maps all of $C_0(\Gamma, \mathfrak{g}^*)$ to 0 by definition. The $j$-cycle space is defined to be

$$Z_j(\Gamma, \mathfrak{g}^*) = \ker \partial_j \subseteq C_j(\Gamma, \mathfrak{g}^*)$$

while the $j$-boundary space is

$$B_j(\Gamma, \mathfrak{g}^*) = \mathrm{im}\, \partial_{j+1} \subseteq C_j(\Gamma, \mathfrak{g}^*)$$

In this simple case, these spaces are, for $\Gamma$,

$$\begin{aligned} Z_0(\Gamma, \mathfrak{g}^*) &= C_0(\Gamma, \mathfrak{g}^*) \\ B_0(\Gamma, \mathfrak{g}^*) &= \operatorname{im} \partial_1 \\ Z_1(\Gamma, \mathfrak{g}^*) &= \ker \partial_1 \\ B_1(\Gamma, \mathfrak{g}^*) &= \{0\} \end{aligned}$$

Because $\Gamma$ is a graph, all the action pertains to the 1-boundary map $\partial_1$. In what follows, we will drop the subscript 1 when no confusion arises.

For $i = 1, 2$, the dual vector spaces of $C_i(\Gamma, \mathfrak{g}^*)$, the vector spaces of real valued linear functions on $C_i(\Gamma, \mathfrak{g}^*)$, are denoted by $C^i(\Gamma, \mathfrak{g}^*)$. We have $C^i(\Gamma, \mathfrak{g}^*) = \mathfrak{g} \otimes C^i(\Gamma, \mathbb{R})$, where $C^i(\Gamma, \mathbb{R})$ is the dual space of $C_i(\Gamma, \mathbb{R})$. An element of $C^i(\Gamma, \mathfrak{g}^*)$ is called an *$i$-cochain*.

A basis for $C^0(\Gamma, \mathbb{R})$ consists of the functions $\boldsymbol{v}^j : C_0(\Gamma, \mathbb{R}) \to \mathbb{R}$, $j = 1, \cdots, n$ defined by

$$\boldsymbol{v}^j(\boldsymbol{v}_i) = \delta_i^j \tag{14}$$

and extended to the rest of $C_0(\Gamma, \mathbb{R})$ by linearity. An element of $C^0(\Gamma, \mathfrak{g}^*)$ can be written as

$$\boldsymbol{\mu} = \sum_{i=1}^{|V(\Gamma)|} \mu_i \otimes \boldsymbol{v}^i \tag{15}$$

where $\mu_i \in \mathfrak{g}$, for $i = 1, \cdots, |V(\Gamma)|$. The value of $\boldsymbol{\mu}$ at $\boldsymbol{x}$ of the form (10) is

$$\boldsymbol{\mu}(\boldsymbol{x}) = \sum_{j=1}^{|V(\Gamma)|} x_j(\mu_j). \tag{16}$$

From this discussion, we distill the following theorem as the first main point of this development.

*Theorem 1:* The space $C^0(\Gamma, \mathfrak{g}^*)$ can be identified with the Lie algebra $\mathfrak{t}$ of the group $\mathcal{T}$.

A basis for $C^1(\Gamma, \mathfrak{g}^*)$ consists of the functions $\boldsymbol{e}^j : C_1(\Gamma, \mathfrak{g}^*) \to \mathfrak{g}^*$, $j = 1, \cdots, m$ defined by

$$\boldsymbol{e}^j(\boldsymbol{e}_i) = \delta_i^j \tag{17}$$

and extended to the rest of $C_1(\Gamma, \mathbb{R})$ by linearity. An element of $C^1(\Gamma, \mathfrak{g}^*)$ can be written as

$$\boldsymbol{\omega} = \sum_{i=1}^{|E(\Gamma)|} \omega_i \otimes \boldsymbol{e}^i \tag{18}$$

where $\omega_i \in \mathfrak{g}$, for $i = 1, \cdots, |E(\Gamma)|$ The *coboundary map* $\partial^* : C^0(\Gamma, \mathfrak{g}^*) \to C^1(\Gamma, \mathfrak{g}^*)$ is the adjoint of $\partial$ defined for $\mu \in C^0(\Gamma, \mathfrak{g}^*)$, $z \in C_1(\Gamma, \mathfrak{g}^*)$

$$\boldsymbol{\mu}(\partial \boldsymbol{z}) = (\partial^* \boldsymbol{\mu})(\boldsymbol{z}) \tag{19}$$

Associated with $\Gamma$ is the *cochain complex* that is the dualization of the chain complex in (13):

$$\{0\} \xrightarrow{\partial_0^*} C^0(\Gamma, \mathfrak{g}^*) \xrightarrow{\partial_1^*} C^1(\Gamma, \mathfrak{g}^*) \xrightarrow{\partial_2^*} \{0\}$$

By duality, the maps $\partial_j^*$ satisfy

$$\operatorname{im} \partial_j^* \subseteq \ker \partial_{j+1}^*$$

The $j$-cocycle space is defined to be

$$Z^j(\Gamma, \mathfrak{g}^*) = \ker \partial_{j+1}^* \subseteq C^j(\Gamma, \mathfrak{g}^*)$$

while the $j$-coboundary space is

$$B^j(\Gamma, \mathfrak{g}^*) = \operatorname{im} \partial_j \subseteq C^j(\Gamma, \mathfrak{g}^*).$$

These spaces are:

$$\begin{aligned} Z^0(\Gamma, \mathfrak{g}^*) &= \ker \partial_1^* \\ B^0(\Gamma, \mathfrak{g}^*) &= \{0\} \\ Z^1(\Gamma, \mathfrak{g}^*) &= C^1(\Gamma, \mathfrak{g}^*) \\ B^1(\Gamma, \mathfrak{g}^*) &= \operatorname{im} \partial_1^* \end{aligned}$$

As in the "chain" case, the simplicity of this situation allows us to make the following identifications

$$\begin{aligned} C^0(\Gamma, \mathfrak{g}^*) &\cong \mathfrak{t} \\ Z^0(\Gamma, \mathfrak{g}^*) &\cong \mathfrak{t}_0 \\ C^0(\Gamma, \mathfrak{g}^*)/Z^0(\Gamma, \mathfrak{g}^*) &\cong \tilde{\mathfrak{t}} \\ B_0(\Gamma, \mathfrak{g}^*) &\cong \tilde{\mathfrak{t}}^* \end{aligned}$$

The second main point of this development requires the introduction of the distributional Fisher information. In this context, (2) becomes, for each edge $e \in E(\Gamma)$, $\mathcal{E}^{\boldsymbol{e}} \in \mathfrak{g}^* \otimes \mathfrak{g}^*$

$$\mathcal{E}^{\boldsymbol{e}} = \int_{\mathcal{A}_{\boldsymbol{e}}} d_g \log p_{\boldsymbol{e}}(g^{-1}\epsilon)|_{\mathrm{id}} \otimes d_g \log p_{\boldsymbol{e}}(g^{-1}\epsilon)|_{\mathrm{id}} p_{\boldsymbol{e}}(\epsilon) d\mu(\epsilon).$$

This gives a pairing on $C^1(\Gamma, \mathfrak{g}^*)$:

$$\langle \sum_{i=1}^{|E(G)|} \omega_i \otimes \boldsymbol{e}^i, \sum_{i=1}^{|E(G)|} \omega_i' \otimes \boldsymbol{e}^i \rangle_{\mathcal{E}} = \sum_{i=1}^{|E(G)|} \mathcal{E}^{\boldsymbol{e}^i}(\omega_i, \omega_i'),$$

in terms of which we can define the map $\mathcal{E} : C^1(\Gamma, \mathfrak{g}^*) \to C_1(\Gamma, \mathfrak{g}^*)$ by

$$\mathcal{E}(\omega) = \sum_{j=1}^{|E(\Gamma)|} \mathcal{E}^{e_j}(\cdot, \omega_j) \otimes \boldsymbol{e}_j$$

The Fisher information at $\mathrm{id} \in \tilde{\mathcal{T}}$ evaluated at $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathfrak{t}$, can be written in terms of the distributional Fisher information on the edges as

$$F_{\mathrm{id}}(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = (\partial \mathcal{E} \partial^*(\boldsymbol{\mu}_2))(\boldsymbol{\mu}_1) \tag{20}$$

The Fisher information at $\mathrm{id} \in \tilde{\mathcal{T}}$ can be regarded as a map from $\tilde{\mathfrak{t}} \to \tilde{\mathfrak{t}}^*$ defined by

$$\hat{F}_{\mathrm{id}}(\boldsymbol{\mu}) = F_{\mathrm{id}}(\cdot, \boldsymbol{\mu})$$

for $\boldsymbol{\mu} \in \tilde{\mathfrak{t}}$. This discussion is summed up in the following theorem.

*Theorem 2:* The structure of the Fisher information is represented by the following commutative diagram

$$\begin{array}{ccc} B_0(\Gamma, \mathfrak{g}^*) & \xleftarrow{\partial} & C_1(\Gamma, \mathfrak{g}^*)/Z_1(\Gamma, \mathfrak{g}^*) \\ {\scriptstyle \hat{F}_{\mathrm{id}}}\big\uparrow & & {\scriptstyle \mathcal{E}}\big\uparrow \\ C^0(\Gamma, \mathfrak{g}^*)/Z^0(\Gamma, \mathfrak{g}^*) & \xrightarrow{\partial^*} & B^1(\Gamma, \mathfrak{g}^*) \end{array}$$

## V. Conclusion

Motivated by registration problems in networks, we have developed mathematical foundations for a theory of gauge-invariant estimation on graphs. In order to accommodate networks whose nodal data are elements of a Lie group, as occurs in many practical problems, we have described the essential elements of information geometry on Lie groups in some generality. We proceeded to describe how the registration problem can be expressed in terms of a connection on the edges of the network graph, thereby framing the problem of registration using noisy data in terms of estimating a gauge transformation that aligns local coordinate systems across the network. We showed how the Fisher Information for this estimation problem depends simultaneously on the geometry of the Lie group on which the measurements are distributed and on algebraic descriptors of the topology of the network graph and explained how it provides a link between the homological chains and cochains for the graph.

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