Invariant Detection and Estimation for MIMO Radar Signals

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Abstract—Motivated primarily by electronic surveillance applications, but also by other potential uses in passive exploitation of radio frequency (RF) signals, this paper considers the problems of detecting the presence of and characterizing a radar transmitter using data collected at a spatially distributed suite of receivers. A characterization of a particular interest is determining the rank of the transmitted signal, which enables discrimination between multiple-input multiple-output (MIMO) and conventional radar transmitters as well as distinguishing between MIMO systems that simultaneously emit different numbers of linearly independent signals from their transmit arrays. In this paper, an invariant posterior distribution for position and signal rank of a MIMO radar emitter is derived based on non-informative prior distribution for the signal parameters. This allows MAP-based detection and signal rank estimation. These estimators are shown to significantly outperform maximum likelihood (ML)/BIC position and rank estimators.

I. INTRODUCTION

MIMO radar concepts have been explored extensively in research literature over the past decade. Among these is the idea of using a transmit array consisting of \( K \geq 2 \) closely-spaced elements that emit linearly independent waveforms. In what follows, Friedlander’s terminology convention is adopted; i.e., the term MIMO radar is reserved for schemes involving spatial waveform diversity in a phased array radar, while multistatic radar is used to mean that transmit and receive components are spatially distributed [1].

This work is motivated primarily by electronic surveillance applications and other passive exploitation applications, where typical goals are to detect and characterize RF signals and to localize their sources using passive means. Specifically, it is assumed that data are collected by a suite of \( M \) spatially distributed receivers with the objectives of detecting and characterizing radar or other RF transmissions. Although the elements in the transmit array may be well separated in terms of the wavelength of the carrier frequency, in the far-field they are essentially co-located on the scale of wavelengths present in the baseband waveform. So, \( K \) linearly independent transmit waveforms will define a \( K \)-dimensional subspace of the space of signals collected across the receiver array.

The problem of detecting a common but unknown signal of known rank \( K \) using data collected at \( M \) spatially distributed receivers has been addressed in recent signal processing literature [2], [3]. A recent paper [4] set forth an approach to this problem that yielded bayesian detectors and rank estimators for the unknown noise variance case. These detectors and estimators are not scale invariant, however. Furthermore, as noted in [4], ML rank estimation is problematic because, without additional regularization, it is intrinsically biased toward overestimating the true signal rank.

This paper addresses these deficiencies by extending the signal rank estimation approaches developed in [3] to the problem of MIMO radar signal characterization. Specifically, an invariant joint posterior distribution for emitter position and rank is derived based on non-informative prior distributions. ML estimators of emitter position and rank, modified by the bayesian information criterion (BIC) penalty, are also developed and compared to the bayesian detector/estimator.

II. PROBLEM AND MATHEMATICAL FORMULATION

Consider a scenario in which a MIMO radar with a \( K \)-element transmit array simultaneously emits \( K \) linearly independent waveforms, \( s(t) = (s_1(t), s_2(t), \ldots, s_K(t))^T \), where each \( s_k(t) \) represents a complex baseband signal. Let \( a(k) = (a_1(k), a_2(k), \ldots, a_K(k))^T \) denote the transmit array steering vector in the direction \( k \). In the far field, the transmitted waveform in the direction \( k \) can be modeled as

\[
y(t) = a(k)^T s(t) e^{\pi i f_c t}
\]

where \( f_c \) is the carrier frequency of the radar and \( \dagger \) denotes hermitian transpose. The signal transmitted by this MIMO radar impinges on \( M > K \) spatially distributed receivers of a surveillance system. The baseband signal collected at the \( m \)-th receiver is

\[
x_m(t) = \alpha_m a(k) s(t - \tau_m) e^{\pi i f_m t + \nu_m(t)},
\]

where \( \alpha_m \in \mathbb{C} \) accounts for path loss and antenna gain for the \( m \)-th receiver and includes the carrier phase. In this expression, \( \tau_m = ||r_m - r_0||/c \) is the travel time from the transmit array at position \( r_0 \) to the \( m \)-th receiver at position \( r_m \) and \( c \) denotes the speed of light. The Doppler shift at the \( m \)-th receiver is

\[
f_m = -f_c (v_m - v_0) \cdot k_m / c,
\]

where \( v_0 \) is the velocity of the transmitter and \( v_m \) the velocity of the receiver. Finally, \( \nu_m(t) \) is additive white gaussian noise at the \( m \)-th receiver, which is assumed to be independent from receiver to receiver.

Suppose there is a MIMO radar transmitter with \( K > 1 \) elements at a position \( r_0 \) and with velocity \( v_0 \). Then, for \( m = 1, \ldots, M \), the received signals are

\[
x_m(t - \tau_m) e^{-\pi i f_m t} = \alpha_m a(k_m) s(t - \tau_m) + \nu_m'(t),
\]

where \( \nu_m'(t) = \nu_m(t - \tau_m) e^{-\pi i f_m t} \). Equation (1) shows that, when suitably adjusted to account for time differences of...
arrival and Doppler shifts, the $M$ received signal components span a $K$-dimensional space defined by a common time-frequency shift of the $K$ transmitted signals. The problem of detecting the presence of an emitter at a given candidate location $r_0$ and velocity $v_0$ is addressed in [3] and further examined here to test whether the emitter is a MIMO (rank $K > 1$) radar versus a conventional (rank $K = 1$) radar. Thus the tests developed here concern the hypotheses:

$H_0$: The collected signal is just receiver noise.

$H_K$: The signal is from a rank-$K$ transmitter at $(r_0, v_0)$, for $K = 1, 2, \ldots, M - 1$.

The following describes how the hypothesis $H_K$ manifests in the test data. Each received signal $x_m(t)$ is shifted to baseband, compensated for time delays and Doppler shifts corresponding to the values of $r_0$ and $v_0$ being considered in the test, and sampled at an appropriate rate to obtain a set of $M$ complex $N$-vectors. These vectors of received data are grouped into an $M \times N$ matrix $X$. Define $S$ to be a $K \times N$ matrix whose rows are orthonormal (i.e., $SS^H = I_K$) and span a $K$-dimensional signal subspace. Let $P = S^H S$. Then $P$ is an $N \times N$ rank-$K$ orthogonal projection matrix. The collection of all rank-$K$ orthogonal projection matrices constitute the Grassmannian $G_{K,N}$, which is a smooth complex manifold of complex dimension $K(N - K)$ [5], [6]. With this notation, the the model of the received data $X$ can be written as

$$X = AS + \nu,$$

where $A$ is a complex $M \times K$ matrix whose $(m,k)$th element $a_{mk} = \alpha_{mk} (k_m)$ is the amplitude of the $k$th component of the signal vector from the $m$th receiver, and the elements of the noise matrix $\nu$ are independent with $\nu_{ij} \sim CN(0, \sigma^2)$. The values of $A$, $S$, and $\sigma^2$ are assumed to be unknown.

Under $H_0$, the joint probability density function (pdf) of $X$ conditioned on $\sigma^2$ is

$$p(X|H_0, \sigma^2) = (\pi \sigma^2)^{-M N} e^{-\frac{1}{2\sigma^2} Tr(W)}$$

where $W = X^H X/N$. Under $H_K$ with $K > 0$, the joint pdf of $X$ conditioned on $A_K, S_K$, and $\sigma^2$ is

$$p(X|H_K, A_K, S_K, \sigma^2) = (\pi \sigma^2)^{-M N} e^{-\frac{1}{2\sigma^2} Tr(W)} e^{-\frac{1}{2\sigma^2} Tr((A_K A_K^H - A_K S_K X^H X S_K^H A_K^H)\sigma^2)}$$

In what follows, the notation $p(X|K)$ to represent $p(X|H_K)$ and $p(X|K = 0)$ for $p(X|H_0)$ will be used.

III. RANK ESTIMATION BASED ON THE BAYESIAN INFORMATION CRITERION

Consider the joint pdf given in (4). Maximizing over the unknown parameters $A_K, S_K$, and $\sigma^2$, yields the ML estimate of $K$ [3].

$$\hat{K} = \arg \min_{K \in \{0, \ldots, M-1\}} \min_{A_K, S_K, \sigma^2} -\log p(X|H_K)$$

$$= \arg \min_{K \in \{0, \ldots, M-1\}} MN \log \left(1 - \frac{\sum_{j=1}^{K} \lambda_j / \text{Tr}(W)}{K(N-K)}\right),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K$ are the eigenvalues of $W$. Note that the non-zero eigenvalues of $W$ are exactly the eigenvalues of the sample-covariance matrix $\hat{R} = XX^H/N$. An unfortunate consequence of this observation is that, as the hypothesized rank $K$ is increased, the hypotheses $H_K$ become increasingly likely and the ML estimate of $K$ defaults to $K = M - 1$ irrespective of the data. Below, it is shown that the Bayesian detector is much better behaved in this regard. Before discussing the Bayesian detector, however, regularization of the ML detector is considered.

The ML estimate of $K$ given in (5) can be improved by introducing a penalty function based on one of the so-called information criteria [7], [8]. With this approach, the estimate (5) is replaced by

$$\hat{K} = \arg \min_{K \in \{0, \ldots, M-1\}} \min_{A_K, S_K, \sigma^2} -\log p(X|H_K) + L(\nu(K), N),$$

where $L(\nu(K), N)$ is a penalty function that depends on the number of parameters $\nu(K)$ and the number of samples $N$. The penalty function for the Bayesian information criterion is $L(\nu(K), N) = \frac{1}{2\sigma^2} \log N$, where $\nu(K)$ denotes the number of undetermined parameters in the likelihood function. In the model (2), $A_K$ consists of $2MK$ real parameters, the Grassmannian $G_{K,N}$ presents $2K(N-K)$ real parameters and there is one real parameter for the noise variance $\sigma^2$ when it is unknown. Therefore, the total number of undetermined parameters is $2MK + 2(N - K) + 1$ in the unknown noise case and $2MK + 2K(N - K)$ if the noise variance is known. The BIC thus augments the log-likelihood as

$$\text{BIC}(K) = MN \log \left(1 - \frac{\sum_{j=1}^{K} \lambda_j / \text{Tr}(W)}{K(N-K)}\right) - (K(M + N) - K^2 + 1/2) \log N.$$

The BIC estimate of rank for a single emitter is the value of $K$ that maximizes (6).

IV. INVARIANT DETECTOR/ESTIMATOR FOR RANK-$K$ MIMO RADAR SIGNALS

In this section, an invariant joint posterior distribution for the rank of a MIMO emitter at position $(r_0, v_0)$ is derived based on non-informative prior distributions. The joint likelihood under $H_K$ given in (4) is invariant under the transformations

$$X \rightarrow \mu UXV, \ A \rightarrow \mu UA, \ S \rightarrow SV, \text{ and } \sigma \rightarrow \mu \sigma,$$

where $U$ and $V$ are any unitary matrices of dimensions $M \times M$ and $N \times N$, respectively, and $\mu > 0$. The priors for the nuisance parameters $A$, $P$, and $\sigma^2$ in the likelihood functions (3) and (4) are taken to be non-informative as possible. As discussed in [9], the prior for $A$ is chosen to be

$$p(A|\sigma^2, \beta^2, K) = (\pi \beta^2 \sigma^2)^{-MK} e^{-\frac{1}{2\sigma^2} Tr(AA^H)},$$

which is an invariant, proper prior that becomes less informative as $\beta^2 \rightarrow \infty$. This choice of prior is in contrast to [4],
where the prior for $A$ was taken to be uniform, leading to a detector/estimator that was not scale invariant.

The prior for $\sigma^2$ is taken as the maximum entropy prior, which is given in [10] as

$$p(\sigma^{-2}|\tau) = \tau^M e^{-\tau M \sigma^{-2}}.$$  

Marginalizing the likelihoods (3) and (4) with respect to the priors given in (7) and (8) yields

$$p(X|K = 0) = \frac{\tau^M \Gamma(\ell) \text{Tr}(\tilde{W})^{-\ell}}{N! \pi^{MN}}$$

and

$$p(X|K, \beta^2, \tau) = \frac{p(X|K = 0) \prod_{i=1}^{K} \left(1 - \alpha \frac{\text{Tr}(\beta W)}{\text{Tr}(W)} \right)^{-\ell}}{d\mu(P)}$$

where $\ell = MN + 1$, $\tilde{W} = W + \frac{M \tau}{2} I_N$, $\alpha = \frac{\beta^2}{\sqrt{2} \sigma^2}$, and $d\mu(P)$ is an invariant measure on the grassmannian $G_{K,N}$.

The marginalization integral over the grassmannian $G_{K,N}$ in (9) can be approximated using Laplace approximation to obtain (see [3] for details),

$$p(X|K, \beta^2, \tau) \approx \frac{p(X|K = 0) \Gamma(K(N-K) - \ell)}{\text{vol}(G_{K,N}) (1 + \beta^2)^{MK}} \prod_{i=1}^{K} \left(\lambda_i - \tilde{\lambda}_{K+j} + \frac{N \gamma}{\ell} \right)^{-1}.$$  

In this expression, $\gamma = (1 - \alpha \sum_{i=1}^{K} \tilde{\lambda}_i)$ with $\tilde{\lambda}_i = \lambda_i/\text{Tr}(\tilde{W})$, and $\text{vol}(G_{K,N})$ denotes the volume of the grassmannian $G_{K,N}$. This volume is given explicitly by

$$\text{vol}(G_{K,N}) = \prod_{n=0}^{K} A_{2n-1} / \prod_{n=1}^{K} A_{2n-1},$$

where $A_n = 2^{n/2} / \Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$ and $\Gamma$ denotes the Gamma function. The posterior distribution for $K$ is then

$$p(K|X, \beta^2, \tau) = \frac{p(K, \beta^2) p(X|K, \beta^2, \tau)}{\sum_{K=0}^{M-1} p(K|\beta^2) p(X|K, \beta^2, \tau)},$$

where $p(K|\beta^2)$ is the prior pdf for $K$ and is chosen to be

$$p(K|\beta^2) = \frac{(1 + \beta^2)^{MK}}{M^{MK}}.$$  

This form of prior for $K$ ensures that, as the prior for $A$ in (7) becomes less informative, the posterior ratios for any two ranks $K$ and $K'$, i.e., $p(K|X)/p(K'|X)$ approach a finite non-zero limit as $\beta^2 \to \infty$ and $\tau \to 0$ yields the posterior distributions

$$p(K = 0|X) = N$$

$$p(K|X) \approx \frac{N^K \Gamma(K(N-K) - \ell)}{\text{vol}(G_{K,N})} \left(\frac{\pi}{\ell} \right)^{K(N-K)} \prod_{i=1}^{K} \prod_{j=1}^{N-K} \left(\tilde{\lambda}_i - \tilde{\lambda}_{K+j} + \frac{N \gamma}{\ell} \right)^{-1}, K = 1, \ldots, M - 1$$

where $\alpha = 1$ and $\gamma = 1 - \sum_{i=1}^{K} \tilde{\lambda}_i$. The normalization $N$ is such that

$$\sum_{K=0}^{M-1} p(K|X) = 1.$$

The MAP estimate of $K$ is

$$\hat{K} = \arg\max_K p(K|X)$$

for which it is unnecessary to compute the normalization.

V. Simulations

In this section, the performance of the MAP detector/estimator for MIMO signals is evaluated through simulation and compared with the BIC detector/estimator. A rank-four MIMO radar simultaneously transmitting four orthogonal waveforms is simulated. The transmitter consists of a four-element uniform linear array with half wavelength spacing at 9.4GHz carrier frequency. The waveform consists four-channel binary phase coded pulses with each pulse has a code consisting of a Hadamard sequence (row of a Hadamard matrix) of length 128 with chip length of 100ns. The four channels have mutually orthogonal Hadamard sequences. The surveillance receiver system consists of six sensors distributed as shown in Figure 1. The total distance spanned by the receivers is 4.725 kilometers and with the most distant receiver at range 4.940km from the radar. The power of the received signals is set relative to the shortest ranges from the radar to the closest sensor, which is 3.904 kilometers and is normalized to unit power. The SNR is defined by $\text{SNR} = -10 \log_{10} \sigma^2$, where $\sigma^2$ is noise power per complex dimension. For simplicity, the transmitters and receivers are assumed to be mutually stationary.

Figures 2 and 3 show the average log posterior pdf and BIC (from (6)) at the true position of the rank-four MIMO emitter. The averages are taken over 5000 trials. These results show that the posterior peaks at the correct emitter rank at 21dB SNR. On the other hand, the expected BIC does not peak at the correct rank until an SNR of 33dB. Note that in order to keep the graphs in these figures uncluttered, only indicative SNRs have been plotted. Figure 4 shows the posterior distribution for emitter rank and position for a single realization at the critical SNR of 21dB. Figure 5 show a similar plot of the BIC, but at the higher BIC-critical SNR of 33dB.

Figures 6 and 7 show results for a rank-five signal. Specifically, Fig. 6 displays average log posterior pdf values for the signal rank at different SNRs and Fig. 7 shows corresponding averaged values of the BIC statistic (6). Other the the signal rank, the configuration and parameters of the simulation are the same as described above. In this case the critical SNRs for correct estimates are 31dB for MAP and 37dB for BIC.

VI. Conclusion

This work considers the problems of detecting the presence of, and characterizing according to rank of the transmitted signal, a radar transmitter using data collected at a spatially distributed suite of receivers. An invariant joint posterior distribution for radar position and rank was derived based on
Fig. 1. Configuration of a transmitter and six distributed sensors.

Fig. 2. Plots of the expected log posterior pdf values for the signal rank at different SNRs.

Fig. 3. Plots of the expected values of BIC (6) for the signal rank at different SNRs.

non-informative prior distributions for the signal parameters. This allows MAP based detection and signal rank estimation. The simulation results show that the invariant posterior pdf or MAP estimator substantially outperforms the BIC estimator; i.e., it is able to estimate the correct rank at considerably lower SNR. In this scenario, the differences are 12dB when the true rank is four and 6dB when the true rank is five.

REFERENCES
Fig. 4. Plots of log of posterior pdf between rank-$K$, $K = 1, \ldots, 5$ for rank-4 (MIMO) phased array radar signals at 21dB SNR.

Fig. 5. Plots of negative values of BIC between rank-$K$, $K = 1, \ldots, 5$ for rank-4 (MIMO) phased array radar signals at 33dB SNR.
Fig. 6. Plots of log of posterior pdf between rank-$K$, $K = 1, \ldots, 5$ for rank-5 (MIMO) phased array radar signals at 31dB SNR.

Fig. 7. Plots of negative values of BIC between rank-$K$, $K = 1, \ldots, 5$ for rank-5 (MIMO) phased array radar signals at 37dB SNR.