ESTIMATION OF SUBSPACE OCCUPANCY

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ABSTRACT

The ability to identify unoccupied resources in the radio spectrum is a key capability for opportunistic user in a cognitive radio environment. This paper draws upon and extends geometrically based ideas in statistical signal processing to develop estimators for the rank and the occupied subspace in a multi-user environment from multiple temporal samples of the signal received at a single antenna. These estimators enable identification of resources (i.e., the orthogonal complement of the occupied subspace) that may be exploitable by an opportunistic user.

***Index Terms*—** Cognitive radio, channel sensing, MAP estimation, rank estimation, multi-channel sensing, MIMO communications

1. INTRODUCTION

The problem of determining whether a signal is present in two or more channels of sensor data has applications in many different fields. The application context where it has been most studied is in defense and security systems, such as radar and sonar, where it pertains to detecting and localizing a target from data collected at multiple geographically distributed sensors. Due to increasing awareness of the need to improve utilization of radio spectrum resources, however, detection methods of this kind have been applied over the past decade in spectrum sensing for cognitive radio in order to ascertain the presence of primary users.

Tests for determining the presence of a common but unknown signal in two or more noisy channels have been studied extensively in connection with passive localization of emitters. Such detectors include those based on the magnitude-square coherence (MSC) estimate [1, 2, 3] and generalized coherence (GC) estimate [4, 5, 6, 7, 8], which are functions of the determinant of a Gram matrix formed from the collected data. The rise of multiple input, multiple output (MIMO) systems in sensing and communications has led to a renewed interest in multiple-channel detection. Motivated in part by MIMO applications, a variety of statistical hypothesis tests have been devised. These tests include generalized likelihood ratio tests (GLRTs), for example the tests derived in [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], locally most powerful invariant tests (LMPITs) as in [21], Bayesian tests such as those set forth in [18, 17, 22, 23], and maximum *a posteriori* (MAP) tests such as those in [22, 24]. Many of these tests exploit an estimate of the spatial covariance matrix of the signals, generally producing a test statistic that is a scalar function of the eigenspectrum of the estimated covariance. Others utilize geometrically based ideas in statistical signal processing that enable rigorously-based statistical tests for spectral occupancy that are optimal under certain assumptions, and often substantially outperform heuristically-based tests even under circumstances where they are not provably optimal.

The primary objective of this paper is to adapt recently introduced estimators for signal subspace rank and the orthogonal projector defining a signal subspace from the multi-received setting to one in which data segments from multiple spatially separated are replaced by temporal data segments from the same antenna. With this formulation in a multi-access communication environment, the receive signal diversity necessary to support rank and subspace estimation is provided by the presence of the same codes in different linear combinations in the respective data segments. The potential efficacy of this scheme for estimating the subspace occupied by primary users, and consequently its unoccupied orthogonal complement, is demonstrated in idealized examples.

The remainder of this paper is organized as follows. Section 2 describes the similarities between testing for a common signal in more classical multichannel detection problems (e.g. radar and sonar) and detecting the presence of one or more signals from primary users. Section 3 briefly explains the derivation of the maximum *a posteriori* (MAP) estimate for rank, as well as an estimate for the orthogonal projection into the occupied subspace. Finally Section 4 shows the application of the MAP rank estimate in a simple CDMA system.

2. MODELS FOR SPECTRUM SENSING AND MULTI-CHANNEL DETECTION

The similarity between testing for occupancy in spectrum sensing and multiple-channel detection and estimation motivated by radar/sonar applications is illustrated by considering models posed by work in these respective regimes. In [25], for example, the signal model for a multi-antenna spectrum sensing scenario is given as

$$\begin{aligned} &H_0: x_m(n) = \eta_m(n) \\ &H_1: x_m(n) = s_m(n) + \eta_m(n), \\ &\quad m = 1, \dots, M \end{aligned} \tag{1}$$

where $M \geq 1$ represents the number of receive antennas, η_m is the noise, and $s_m(n) = \sum_{k=1}^{K} \sum_{l=0}^{q_{mk}} h_{mk}(l) \tilde{s}_k(n-l)$ is the signal received by antenna m. In this expression, K is the number of primary user/antenna signals, $\tilde{s}_k(n)$ denotes the transmitted signal from primary user/antenna k, $h_{mk}(l)$ represents the propagation channel from user k to receiver m, and q_{mk} is the channel order. The noise samples $\eta_m(n)$ are assumed iid for n and m, and the signal, noise and channel coefficients are assumed to be real. In [18], the system model to detect the presence of a rank-K emitter using $M > K$ spatially distributed sensors is given as

$$\begin{aligned} &H_0 : \mathbf{X} = \boldsymbol{\nu} \\ &H_1 : \mathbf{X} = \mathbf{A}\mathbf{S} + \boldsymbol{\nu}, \end{aligned} \tag{2}$$

where a K-dimensional signal subspace is defined by an unknown $N \times K$ complex matrix $\mathbf{S}$ whose columns are orthonormal vectors in $\mathbb{C}^N$, and the element a_{km} of the unknown $K \times M$ complex matrix $\mathbf{A}$ is the complex amplitude of the component of the signal received at sensor m and in the subspace corresponding to the k^{th} column of $\mathbf{S}$. The noise $\boldsymbol{\nu}$ is assumed to be normally distributed with zero mean and is also assumed to be spatially and temporally white with covariance matrix $\sigma^2 \mathbb{I}_{NM}$.

Utilizing the geometrical nature of tests developed in [18, 22] and related work, estimators of pertinent signal structure, such as rank, can be found. With this insight, one can consider estimating the unoccupied subspace in settings where multiple access is not based on frequency division, opening the possibility of opportunistic use of unoccupied communication resources that are not necessarily defined by spectral bands.

3. MATHEMATICAL FORMULATION

3.1. MAP Estimate for Rank

The MAP estimate for signal rank assuming unknown noise variance was calculated in [22], The main steps of the derivation are summarized here, and the result is subsequently applied to determine the number of users in a simple CDMA system. The problem definition for estimating rank becomes a multi-hypothesis modified from (2)

$$\begin{aligned} &H_0 : \mathbf{X} = \boldsymbol{\nu} \\ &H_K : \mathbf{X} = \mathbf{A}_K \mathbf{S}_K + \boldsymbol{\nu}, \end{aligned} \tag{3}$$

where the columns of S_K define a K-dimensional subspace of $\mathbb{C}^N$ for $K = 1, \dots, M-1$.

A posterior distribution for K given the data $\mathbf{X}$ was found by selecting proper priors for the nuisance parameters that are as non-informative as possible and, like the likelihood function under H_K, are invariant under the transformations

$$\mathbf{X} \to \mu \mathbf{U}\mathbf{X}\mathbf{L},\ \mathbf{A} \to \mu \mathbf{U}\mathbf{A},\ \mathbf{S} \to \mathbf{S}\mathbf{L},\ \text{and}\ \sigma \to \mu\sigma \tag{4}$$

where $\mathbf{U}$ and $\mathbf{L}$ are unitary matrices of dimensions $M \times M$ and $N \times N$ respectively, and $\mu > 0$. In order to obtain a non-redundant model parameterization, a rank-K orthogonal projection matrix $\mathbf{P}$ is defined as $\mathbf{P} = \mathbf{S}^\dagger \mathbf{S}$, where it is possible to assign a unique $\mathbf{S}$ to each $\mathbf{P}$. The prior for $\mathbf{A}$ is chosen to be

$$p\left(\mathbf{A} | K, \sigma^2, \beta^2\right) = \left(\pi \beta^2 \sigma^2\right)^{-MK} e^{\frac{1}{\beta^2 \sigma^2} \mathrm{Tr}(\mathbf{A}\mathbf{A}^\dagger)}$$

which becomes less informative as $\beta^2 \to \infty$. The prior for σ^2 is taken to be the maximum entropy prior

$$p\left(\sigma^{-2} | \tau\right) = \tau^M e^{-\tau M \sigma^{-2}}$$

which becomes less informative as $\tau \to 0$. Additionally, the prior for K is

$$p(K|\beta^2) = \frac{\left(1+\beta^2\right)^{MK}}{\sum_{K=0}^{M-1} \left(1+\beta^2\right)^{MK}}$$

which ensures that as the prior for $\mathbf{A}$ becomes less informative, the posterior ratios for any two ranks K and K' approaches a finite non-zero limit. Otherwise, the hypothesis H_K with the smallest value of K would dominate regardless of the data. The prior for $\mathbf{P}$ is taken as the normalized invariant measure on the complex Grassmannian $G_{K,N}$, which is a $K(N-K)$-dimensional smooth manifold comprised of all rank-K orthogonal projection matrices $\mathbf{P}$. The normalized invariant measure is obtained by parameterizing $\mathbf{P}$ in terms of local coordinates on $G_{K,N}$ is found to be

$$d\mu(\mathbf{P}) = \frac{1}{\mathrm{vol}(G_{K,N})} \det\left(\mathbb{I}_K - \mathbf{Z}^\dagger \mathbf{Z}\right)^{-N} d\mathbf{Z}$$

where

$$d\mathbf{Z} = \prod_{i=1}^{N-K} \prod_{j=1}^{K} d\mathrm{Re}(z_{ij}) d\mathrm{Im}(z_{ij})$$

and the volume of the Grassmannian is

$$\mathrm{vol}(G_{K,N}) = \frac{\prod_{n=N-K+1}^{N} A_{2n-1}}{\prod_{n=1}^{K} A_{2n-1}}$$

where $A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the unit sphere in $\mathbb{R}^n$, and Γ denotes the gamma function. As $\beta^2 \to \infty$ and $\tau \to 0$, the marginalized likelihoods can be approximated using Laplace approximation and matrix identities, and the posterior distributions are found to be

$$p(K = 0 \,|\, \mathbf{X}) = C$$

and

$$p(K|\mathbf{X}) = C \frac{1}{\mathrm{vol}(G_{K,N})} \left(\frac{\pi}{p}\right)^{K(N-K)} \gamma^{K(N-K)-p} \times \prod_{i=1}^{K} \prod_{j=1}^{N-K} \left(\tilde{\lambda}_i - \tilde{\lambda}_{K+j} + \frac{N\gamma}{p}\right)^{-1} \quad (5)$$

for $K = 1, \ldots, M-1$, where $p = M(N-K)+1$, $\gamma = (1 - \sum_{i=1}^{K} \tilde{\lambda}_i)$, $\tilde{\lambda}_i = \frac{\lambda_i}{\mathrm{Tr}(\mathbf{W})}$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ are the eigenvalues of the $N \times N$ matrix $\mathbf{W} = \frac{1}{N}\mathbf{X}^\dagger \mathbf{X}$.

The MAP estimate of K is then given by

$$\hat{K} = \arg\max_{K} p(K|\mathbf{X}) \quad (6)$$

for which the computation of C is unnecessary as it is constant for all values of K.

3.2. Estimate for Orthogonal Projector

The likelihood function corresponding to the H_1 signal model (2) with known noise variance σ^2 and signal rank K is

$$p(\mathbf{X}|H_1, \mathbf{A}, \mathbf{S}, \sigma^2) = (\pi\sigma^2)^{-MN} e^{-\frac{N}{\sigma^2}\mathrm{Tr}(\mathbf{W})} \times e^{-\frac{1}{\sigma^2}\mathrm{Tr}((\mathbf{A}-\mathbf{X}\mathbf{S}^\dagger)(\mathbf{A}-\mathbf{X}\mathbf{S}^\dagger)^\dagger - \mathbf{X}\mathbf{S}^\dagger\mathbf{S}\mathbf{X}^\dagger)}$$

Maximizing this likelihood function with respect to $\mathbf{A}$ gives the estimate $\hat{\mathbf{A}} = \mathbf{X}\mathbf{S}^\dagger$. Then, substituting $\hat{\mathbf{A}}$ for $\mathbf{A}$ and using the Schur-Horn theorem to maximize over $\mathbf{P} = \mathbf{S}^\dagger\mathbf{S} \in G_{K,N}$ as in [17] yields the estimate

$$\hat{\mathbf{P}} = \sum_{k=1}^{K} \mathbf{v}_k \mathbf{v}_k^\dagger \quad (7)$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_K$ are the unit-norm eigenvectors of $\mathbf{W}$ corresponding to its K largest eigenvalues. An estimate for the occupied subspace is uniquely specified by the estimated projector $\hat{\mathbf{P}}$.

Note that the K value in the estimate of the projector is the value of the actual rank. If the rank were unknown, the estimate for K, such as $\hat{K}$ given in (6), would have to be used. In such a case, the accuracy of the estimate of the orthogonal projector of subspace would be dependent upon the accuracy of the estimate of the rank.

4. SIMULATIONS

To demonstrate the performance of this approach, a small-scale CDMA system with K users was simulated. In this simulation, K random QPSK sequences are encoded on K PN sequences (codes) of length N. The K codes, normalized to have unit norm, are the rows of the matrix $\mathbf{S}_{K\times N}$ in (3). The m^{th} row of the matrix $\mathbf{A}_{M\times K}$ corresponds to the m^{th} measurement epoch, and its elements are determined by the QPSK sequence elements belonging to each of the K users during that epoch, with $K < M < N$. The noise is zero mean white complex Gaussian, $\boldsymbol{\nu}_{M\times N}$.

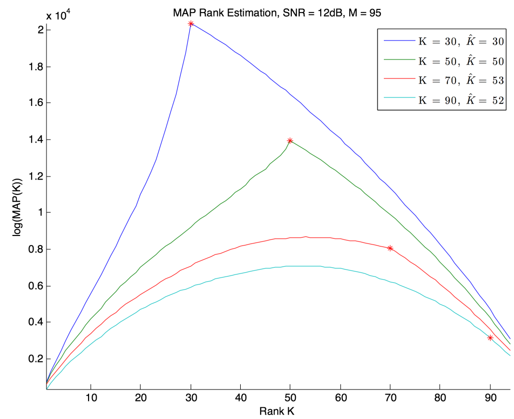

Fig. 1. MAP estimate results for $M = 95$, $N = 100$, $K = 30, 50, 70, 90$ for the blue, green, red and cyan lines respectively, with an SNR at the receiver of 12dB. The red asterisk represents the point where the true rank falls on the MAP. Each line represents a single iteration.

With this set-up, the posterior distribution given in (5) was calculated for $K = 1, \ldots, M-1$, without exploiting knowledge of $\mathbf{S}$, $\mathbf{A}$, or $\boldsymbol{\nu}$ beyond the general structures just described. The value of K giving maximum posterior probability was taken as the estimate of the number of active users.

It can be seen in Figure 1 that when the actual value of K is well below the limit of $M-1$, the MAP estimator does an excellent job correctly estimating the correct rank. However, as the value of the actual number of users K approaches the maximum value, the performance degrades markedly.

By increasing the SNR, the rank can be correctly estimated closer to the limit imposed by the number of data segments used. This can be seen by keeping the same values for K, M and N, but increasing the SNR at the receiver to 24dB. It can be seen in Figure 2 that the estimate is improved closer to the limit. At a certain point, however, the number of primary users becomes sufficiently high that it is not estimated correctly regardless of the SNR. When this point is reached, it is only possible to obtain a correct estimate by increasing the number of (temporal) data segments, and correspondingly the length of the codes. As seen in Figure 3, it is then possible to obtain a correct estimate. However, as M and N grow, the computation time does as well.

By using either the known or estimated number of users, K, the estimate of the orthogonal projection into the occupied subspace from (7) can be calculated. From the estimated orthogonal projection, an estimate for the occupied subspace can be uniquely specified, similarly an estimate for the unoccupied subspace can be specified. The ability to obtain this subspace is useful because it enables codes for the secondary

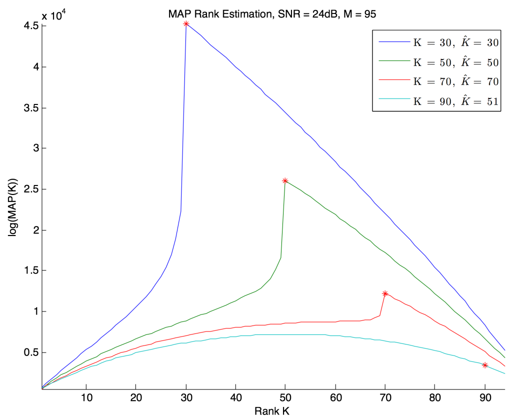

Fig. 2. MAP estimate results for $M = 95$, $N = 100$, $K = 30, 50, 70, 90$ for the blue, green, red and cyan lines respectively, with an SNR at the receiver of 24dB. The red asterisk represents the point where the true rank falls on the MAP. Each line represents a single iteration.

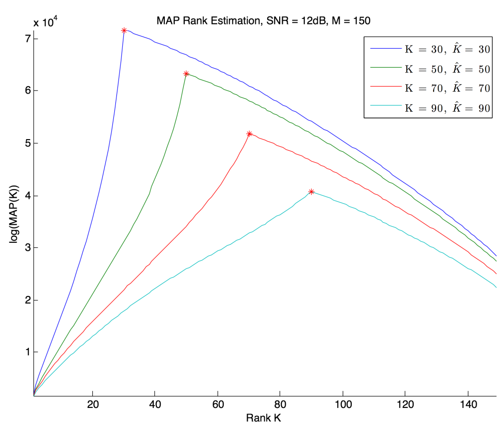

Fig. 3. MAP estimate results for $M = 150$, $N = 200$, $K = 30, 50, 70, 90$ for the blue, green, red and cyan lines respectively, with an SNR at the receiver of 12dB. The red asterisk represents the point where the true rank falls on the MAP. Each line represents a single iteration.

users to be generated that fall in in the unoccupied subspace. If a correct estimate is obtained, these codes will be orthogonal to the codes of the primary users, ideally resulting in the secondary users inflicting no interference on the primary users.

5. CONCLUSION

Motivated by spectrum sensing applications where the allocation of communication resources may not be defined by spectral bands, an example was completed using the MAP estimator for rank to predict the number of users in a simple CDMA system. It was seen that the detector worked well at estimating the number of users under certain parameters, mainly that the signal had a high SNR and that the values were well within their limits. However, as the number of users increases, the number of time epochs and correspondingly, the length of the codes must increase as well to obtain adequate performance. Additionally, an estimator for the orthogonal projector of the occupied subspace was shown. From this projector, it is possible to determine the associated occupied subspace, and by extension the orthogonal complement of the occupied subspace. By selecting codes known to be in the unoccupied subspace, and thus orthogonal to the codes in use, it is possible to reduce the interference caused by secondary users.

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