MULTIPLE-CHANNEL DETECTION OF SIGNALS HAVING KNOWN RANK

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ABSTRACT

Bayesian and generalized likelihood ratio tests are derived for the detection of a common unknown signal of known rank K in $M > K$ independent channels of white gaussian noise. The cases of known and unknown noise variance are both treated. These derivations encompass the development of explicit expressions for an invariant measure on the grassmannian manifold of K-dimensional subspaces of complex N-dimensional space and parameterization of this manifold to enable the calculation of the necessary marginalization integrals. Performance of the detectors is compared by simulation.

***Index Terms*—** Multiple-channel detection, Bayesian detection, GLRT, Grassmannian, Coherence, Known-rank signal

1. INTRODUCTION

Statistical tests for correlation among collections of $M \geq 2$ random vectors have been extensively used in sensor signal processing. With $M = 2$, for example, the magnitude-squared coherence (MSC) estimate has a significant history in connection with detecting the presence of a common but unknown signal in two noisy channels [1]. Motivated in large part by its utility in time delay estimation for such applications as passive sonar and non-invasive monitoring of mechanical systems, invariance properties of the MSC estimate and its optimality and performance as a detection statistic under various assumptions were studied in detail between the 1970s and 1990s (see, e.g., [2, 3, 4] and the tutorial and papers in [5]). Some attention was given simultaneously to the situation with $M > 2$, initially building from estimators of multiple coherence [6]. Interest in this setting increased in the 1980s as the ability to aggregate data from multiple geographically distributed sensors became more prevalent in application systems. In 1988, the generalized coherence (GC) estimate was introduced in [7], and its invariances and performance as a multiple-channel detection statistic were studied extensively during the 1990s [8, 9, 10, 11].

Recent interest in MIMO systems within both sensing and communications contexts has precipitated new veins of work in multiple-channel detection. In [12], a generalized likelihood ratio test (GLRT) for spatial correlation among a collection of complex circular Gaussian signals with unknown arbitrary covariance matrices was derived. GLRTs and locally most powerful invariant tests for vector-valued random processes with covariance matrices of known rank were developed in [13, 14], primarily in the context of MIMO communications applications. A bayesian test for diagonal covariance matrix versus arbitrary non-diagonal covariance matrix for a zero-mean complex gaussian $M \times N$ matrix was derived in [15], and the GC estimate was shown to be a sufficient statistic for this test.

This paper examines the situation in which the putative unknown signal component in M channels of additive white gaussian noise is known to have rank K with $K < M$. A specific model for this situation is set forth in Section 2. Subsequently, two GLRTs are derived: one for the case in which the noise variance is known and the other for the case in which it is not known. The model used here is different from the one employed in [13] in that it assumes an unknown K-dimensional signal subspace that does not vary over time rather than a rank-K covariance matrix. The derivation here (as well as the form of the GLR) is correspondingly different that the one in [13], and it is instructive in setting the stage for obtaining Bayesian tests in Section 4. The crucial step in deriving the Bayesian tests, one for known noise variance and one for unknown noise variance, is marginalization of likelihood functions by integration over the grassmannian manifold $\mathbf{G}_{K,N}$ of all K-dimensional subspaces in $\mathbb{C}^N$ with respect to a suitably normalized invariant measure. The invariant measure and the parameterization of $\mathbf{G}_{K,N}$ used to evaluate the marginalization integrals are discussed in Appendix A.

2. MODEL AND BACKGROUND

Consider a detection scenario in which the presence of an unknown signal of known rank K is to be detected using a suite of $M > K$ spatially distributed sensors. Received data at each sensor is assumed to have been filtered to a band of interest, suitably sampled, and appropriately adjusted in time delay and Doppler to correspond to a physical state of interest for the test. This pre-processing provides M complex vectors, each of length N, which serve as the data for the detector. Precisely, the data is modeled as a $M \times N$ data matrix

$$X = AS + \nu,$$

whose elements x_{mn} represents samples of the noisy signal collected at the m^{th} sensor at time n. The K-dimensional signal subspace is defined by $S \in \mathbb{C}^{K \times N}$, whose rows are orthonormal vectors in $\mathbb{C}^N$, and the element a_{mk} of the matrix $A \in \mathbb{C}^{M \times K}$ is the complex amplitude of the component of the signal received at sensor m and in the subspace corresponding to the k^{th} row of S. Both A and S are unknown, except for the properties just described; in Section 4, however, a prior distribution on A will be assumed to derive a bayesian test. The noise matrix ν is normally distributed with zero mean and is both spatially and temporally white; i.e., its $NM \times NM$ covariance matrix is $\sigma^2 I_{NM}$.

The following sections address the detection problem

$$\begin{aligned} &H_0 : X \sim \mathcal{CN}(0, \sigma^2 I) \\ &H_1 : X \sim \mathcal{CN}(AS, \sigma^2 I), \text{for some } S \text{ and some non-zero } A \end{aligned} \tag{1}$$

Under H_0, the joint probability density function of X conditioned on σ^2 is

$$p(X|H_0,\sigma^2) = (\pi\sigma^2)^{-MN} e^{-\frac{N}{\sigma^2}\mathrm{Tr}(W)}, \tag{2}$$

Under H_1 the joint pdf of X conditioned on σ^2, A, and S is

$$\begin{aligned} &p(X|H_1 A, S, \sigma^2) \\ &= (\pi\sigma^2)^{-MN} e^{-\frac{1}{\sigma^2}\mathrm{Tr}(X-AS)(X-AS)^\dagger} \\ &= (\pi\sigma^2)^{-MN} e^{-\frac{N}{\sigma^2}\mathrm{Tr}(W)} e^{-\frac{1}{\sigma^2}\mathrm{Tr}((A-XS^\dagger)(A-XS^\dagger)^\dagger - XS^\dagger S X^\dagger)} \end{aligned} \tag{3}$$

where $W = \frac{1}{N}X^\dagger X$. Note that the non-zero eigenvalues of W are exactly the eigenvalues of the sample-covariance matrix $\frac{1}{N}XX^\dagger$.

3. GLRT FOR RANK-K SIGNALS

This section derives the generalized likelihood ratio test (GLRT) statistic for the detection problem (1). Recall that the signal rank K is assumed known. The GLRT is obtained by considering the ratio of maximal values of the joint likelihood functions of the unknown parameters under the hypotheses H_0 and H_1 as follows.

$$\frac{\max p(X|A,S,\sigma^2)}{\max p(X|\sigma^2)} \gtrless_{H_0}^{H_1} \gamma$$

3.1. GLRT with known noise variance

If the noise variance σ^2 is known, it can be omitted from the maximization, and maximizing the likelihood function (3) with respect to A yields the estimate $\hat{A} = XS^\dagger$. Substituting this estimate into the likelihood ratio gives

$$\frac{\max_{A,S} p(X|A,S,\sigma^2)}{p(X|\sigma^2)} = e^{\frac{1}{\sigma^2}\mathrm{Tr}(XS^\dagger S X^\dagger)} = e^{\frac{N}{\sigma^2}\mathrm{Tr}(WP)} \tag{4}$$

where $P = S^\dagger S$ is an orthogonal projection into the K-dimensional subspace defined by the rows of S. The space of all K-dimensional subspaces of an N-dimensional vector space V over the complex numbers is the grassmannian $\mathbf{G}_{K,N}$. The points in $\mathbf{G}_{K,N}$ are in one-to-one correspondence with the orthogonal projections P on V with $\mathrm{Tr}(P) = K$. Hence maximizing the generalized likelihood ratio (4) over P is achieved by maximizing $\mathrm{Tr}(WP)$ over the grassmannian

$$\arg\max_{P\in\mathbf{G}_{K,N}} \mathrm{Tr}(DU^\dagger PU) = \arg\max_{P\in\mathbf{G}_{K,N}} \mathrm{Tr}(DP')$$

where $W = UDU^\dagger$, U is an $N \times N$ unitary matrix, and D is an $N \times N$ diagonal matrix with non-increasing eigenvalues $d_D = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Obviously $\mathrm{Tr}(W) = \mathrm{Tr}(D) = \lambda_1 + \cdots + \lambda_N$. Note that P' is still an orthogonal projector, and it is notationally convenient to rename P' to P. The Schur-Horn theorem [16] states that, if $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is a vector of non-increasing eigenvalues of a Hermitian matrix H, then the vector d_H of diagonal elements of H is in the convex hull of all permutations $\rho(\Lambda)$ of Λ; i.e.,

$$d_H = \sum_\rho a_\rho \rho(\Lambda), \quad 0 \le a_\rho \le 1, \sum_\rho a_\rho = 1$$

Since P is a rank-K orthogonal projection, its eigenvalues are 1 with multiplicity K and 0 with multiplicity $(N-K)$; i.e., its vector of eigenvalues is $d_P = (\mathbb{1}_K, \mathbf{0}_{N-K})$ where $\mathbb{1}_n$ and $\mathbf{0}_n$ denote $1 \times n$ vectors of ones and zeros, respectively. Thus

$$\mathrm{Tr}(DP) = d_D d_P^\mathsf{T} = \sum_\rho a_\rho d_D [\rho(\mathbb{1}_K, \mathbf{0}_{N-K})]^\mathsf{T} \tag{5}$$

The unique maximum of the right-hand side of (5) occurs at $\rho = I_N$; i.e., $\mathrm{Tr}(DP)$ is maximized when

$$P = P_0 = \begin{pmatrix} I_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Substituting P_0 into (4) gives the generalized likelihood ratio when the noise variance is known as

$$\mathrm{GLR} = e^{\frac{N}{\sigma^2}\mathrm{Tr}(DP_0)} = e^{\frac{N}{\sigma^2}\sum_{i=1}^K \lambda_i}$$

3.2. GLRT with unknown noise variance

If σ^2 is unknown, the likelihood functions under H_1 and H_0 must also be maximized with respect to σ^2. The ML estimates are $\hat{\sigma^2} = \mathrm{Tr}(D)/M$ under H_0 and $\hat{\sigma^2} = (\mathrm{Tr}(D) - \mathrm{Tr}(DP))/M$ under H_1. Substituting the estimates of A and σ^2 into the likelihoods (3) and (2), yields

$$\frac{\max_{A,S,\sigma^2} p(X|A,S,\sigma^2)}{\max_{\sigma^2} p(X|\sigma^2)} = (1 - \mathrm{Tr}(DP)/\mathrm{Tr}(D))^{-MN} \tag{6}$$

As in section 3.1, maximizing (6) over P is accomplished by maximizing $\mathrm{Tr}(DP)$ over $\mathbf{G}_{K,N}$. Substituting P_0 from this maximization into (6) yields the GLR for the unknown noise variance case as

$$\mathrm{GLR} = \left(1 - \frac{\sum_{i=1}^K \lambda_i}{\sum_{i=1}^N \lambda_i}\right)^{-MN}. \tag{7}$$

4. BAYESIAN APPROACH FOR RANK K DETECTION

In this section, a bayesian approach is taken to address the detection problem (1). With this approach, instead of formulating ML estimates of the nuisance parameters A, S, and σ^2, each parameter is marginalized out of the likelihood functions in which it appears by integration with respect to a corresponding prior probability distribution. The essential element of this procedure is computation of the integral

$$p(X|H_1) = \int p(X|A,S,\sigma^2) p(A) p(S) p(\sigma^{-2})\, dA\, dS\, d(\sigma^{-2})$$

First, the following prior pdf for A is chosen to accommodate the marginalization:

$$p(A) = (\pi\sigma_a^2)^{-MK} e^{-\frac{1}{\sigma_a^2}\mathrm{Tr}(AA^\dagger)}$$

This prior yields the marginalised likelihood

$$p(X|S,\sigma^2) = \frac{\sigma^{-2MN}}{\pi^{MN}(1+\beta^2)^{MK}} e^{-\frac{N}{\sigma^2}(\mathrm{Tr}(D) - \alpha\mathrm{Tr}(DP))} \tag{8}$$

where $\beta^2 = \sigma_a^2/\sigma^2$ and $\alpha = \beta^2/(1+\beta^2)$. Note that a uniform non-informative prior cannot be used here as it leads to a non-normalizable distribution for X.

4.1. Bayesian detector with known noise variance

If σ^2 is known, the marginalized likelihood ratio reduces to

$$\frac{p(X|S,\sigma^2)}{p(X|\sigma^2)} = \frac{1}{(1+\beta^2)^{MK}} e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(DP)} \tag{9}$$

P is marginalized by integrating (9) over the grassmannian with respect to an invariant measure on $\mathbf{G}_{K,N}$; i.e.,

$$\frac{1}{(1+\beta^2)^{MK}}\int_{P\in\mathbf{G}_{K,N}} e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(DP)}d\mu(P) \qquad (10)$$

In order to compute the integral (10), it is necessary to parametrize P in local coordinates on $G_{K,N}$. Using the results in Appendix A, (10) becomes

$$\frac{1}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}\int_{Z\in\mathbb{C}^{(N-K)\times K}} e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(DP_z)}\det(I_K+Z^\dagger Z)^{-N}dZ \qquad (11)$$

where P_z as given in (19) and $dZ=\prod_{i=1}^{N-K}\prod_{j=1}^{K} d\Re(z_{ij})d\Im(z_{ij})$. Rewriting (11) as

$$\frac{1}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}\int_Z e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(DP_z)}e^{-N\log\det(I_K+Z^\dagger Z)}dZ \qquad (12)$$

this integral can be evaluated using the Laplace approximation [17] with the following identities:

$$(I_K+Z^\dagger Z)^{-1} = I_K - ZZ^\dagger + (Z^\dagger Z)^2 + \cdots \approx I_K - Z^\dagger Z$$

$$\log\det(I_K+Z^\dagger Z) = \mathrm{Tr}(Z^\dagger Z - \frac{1}{2}(Z^\dagger Z)^2 + \dots) \approx \mathrm{Tr}(Z^\dagger Z)$$

$$\mathrm{Tr}(DP_z) = \mathrm{Tr}(D_K(I_K+Z^\dagger Z)^{-1}) + \mathrm{Tr}(D_{N-K}Z(I_K+Z^\dagger Z)^{-1}Z^\dagger)$$
$$\approx \mathrm{Tr}(D_K) - \mathrm{Tr}(D_K Z^\dagger Z) + \mathrm{Tr}(D_{N-K}ZZ^\dagger)$$

where D_K and D_{N-K} are diagonal matrices with the first K and the last $N-K$ eigenvalues of D, respectively. The integral (12) becomes

$$\eta e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(D_K)}\int e^{-\frac{N\alpha}{\sigma^2}\left(\mathrm{Tr}\left(\left(D_K+\frac{\sigma^2}{\alpha}I_K\right)Z^\dagger Z\right)-\mathrm{Tr}\left(D_{N-K}ZZ^\dagger\right)\right)}dZ$$

$$= \eta e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(D_K)}\int e^{-\frac{N\alpha}{\sigma^2}\left(\sum_{i=1}^{N-K}\sum_{j=1}^{K}(\lambda_j-\lambda_{K+i}+\frac{\sigma^2}{\alpha})\right)|z_{ij}|^2}dz_{ij}$$

$$= \eta e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(D_K)}\prod_{i=1}^{N-K}\prod_{j=1}^{K}\int e^{\frac{N\alpha}{\sigma^2}\left(\lambda_j-\lambda_{K+i}+\frac{\sigma^2}{\alpha}\right)|z_{ij}|^2}dz_{ij}$$

$$= \eta e^{\frac{N\alpha}{\sigma^2}\mathrm{Tr}(D_K)}\prod_{i=1}^{N-K}\prod_{j=1}^{K}\frac{\pi\sigma^2}{N\alpha(\lambda_j-\lambda_{K+i}+\frac{\sigma^2}{\alpha})}$$

where $\eta = (1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})$. The likelihood ratio when σ^2 is known is thus

$$Qe^{\frac{N\alpha}{\sigma^2}\sum_{i=1}^{K}\lambda_i}\prod_{i=1}^{N-K}\prod_{j=1}^{K}\frac{1}{\lambda_j-\lambda_{K+i}+\frac{\sigma^2}{\alpha}}$$

where $Q = (\frac{\pi\sigma^2}{N\alpha})^{K(N-K)}\frac{1}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}$.

4.2. Unknown noise variance

When σ^2 is unknown, σ^2 may be marginalized by integrating (8) with respect to an entropy prior as proposed in [15],

$$p(\sigma^{-2}I_M) = \tau^{Mq}\Gamma^{-M}(q)\sigma^{-2M(q-1)}e^{-\tau M\sigma^{-2}} \qquad (13)$$

Thus under H_1, the following posterior distribution is obtained:

$$\frac{\eta}{(1+\beta^2)^{MK}}\int\sigma^{-2(p-1)}e^{-\frac{N}{\sigma^2}\left(\mathrm{Tr}(D)-\alpha\mathrm{Tr}(DP)+\frac{\tau M}{N}\right)}d\sigma^{-2}$$

where $p=M(N+q-1)+1$ and $\eta = \tau^{Mq}/\pi^{MN}\Gamma^M(q)$. Integration yields

$$p(X|H_1,S) = \frac{\eta\Gamma(p)}{(1+\beta^2)^{MK}}(N(\mathrm{Tr}(D)-\alpha\mathrm{Tr}(DP)+\frac{\tau M}{N}))^{-p}$$

Similarly,

$$p(X|H_0) = \eta\Gamma(p)(N(\mathrm{Tr}(D)+\frac{\tau M}{N}))^{-p}$$

where $q > 1-N-1/M$. A minimal prior assumption is to take $q=1$. As $\tau\to 0$, the marginalized likelihood ratio is

$$\frac{p(X|H_1,S)}{p(X|H_0)} = \frac{1}{(1+\beta^2)^{MK}}(1-\alpha\mathrm{Tr}(\tilde{D}P))^{-p} \qquad (14)$$

where the notation $\tilde{D} = D/\mathrm{Tr}(D)$ is introduced. Marginalizing P in (14) is achieved by integrating with respect to invariant measure, taken to be the prior for P, over $\mathbf{G}_{K,N}$; i.e.,

$$\frac{p(X|H_1)}{p(X|H_0)} = \frac{1}{(1+\beta^2)^{MK}}\int_{P\in\mathbf{G}_{K,N}}(1-\alpha\mathrm{Tr}(\tilde{D}P))^{-p}d\mu(P)$$

Again, through parametrization of the grassmannian, the integral becomes

$$\frac{1}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}\int_Z e^{-p\log(1-\alpha\mathrm{Tr}(\tilde{D}P_Z))-N\log\det(I_K+Z^\dagger Z)}dZ$$

which can be approximated using the Laplace approximation. Applying the following identity (up to second order in Z)

$$\log(1-\alpha\mathrm{Tr}(\tilde{D}P_Z)) \approx \log(1-\alpha\mathrm{Tr}(\tilde{D}_K)) + \gamma\left(\mathrm{Tr}(\tilde{D}_K Z^\dagger Z) - \mathrm{Tr}(\tilde{D}_{N-K}ZZ^\dagger)\right)$$

where $\gamma = \alpha/(1-\alpha\mathrm{Tr}(\tilde{D}_K))$. The integral becomes

$$\frac{(1-\alpha\mathrm{Tr}(\tilde{D}_K))^{-p}}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}\int e^{-p\gamma\left(\mathrm{Tr}((\tilde{D}_K+\delta I_K)Z^\dagger Z)-\mathrm{Tr}(\tilde{D}_{N-K}ZZ^\dagger)\right)}dZ$$

$$= \frac{(1-\alpha\mathrm{Tr}(\tilde{D}_K))^{-p}}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}\int e^{-p\gamma\left(\sum_{i=1}^{N-K}\sum_{j=1}^{K}(\lambda_j-\lambda_{K+i}+\delta)\right)|z_{ij}|^2}dz_{ij}$$

$$= \frac{(1-\alpha\mathrm{Tr}(D_K))^{-p}}{(1+\beta^2)^{MK}\mathrm{vol}(G_{K,N})}\prod_{i=1}^{N-K}\prod_{j=1}^{K}\frac{\pi}{p\gamma(\lambda_j-\lambda_{K+i}+\delta)}$$

where $\delta = N/p\gamma$. The likelihood ratio is thus

$$\frac{p(X|H_1)}{p(X|H_0)} = Q\left(1-\alpha\frac{\sum_{i=1}^{K}\lambda_i}{\sum_{i=1}^{N}\lambda_i}\right)^{K(N-K)-p}\prod_{i=1}^{N-K}\prod_{j=1}^{K}\frac{1}{\lambda_j-\lambda_{K+i}+\delta}$$

where $Q = (\pi/p\alpha)^{K(N-K)}/((1+\beta^2)^{MK}\mathrm{vol}(G_{K,N}))$ and recall $p = MN+1$.

5. PERFORMANCE RESULTS

Ideally, the bayesian detectors developed here should be evaluated in terms of criteria such as expected loss minimization or probability of error. It is convenient here to compare with the GLRT in terms of ROC curves in a particular situation, leaving a more through analysis to a future paper. Figure 1 shows the ROC curve for GLRT and bayesian detectors with known and unknown noise.

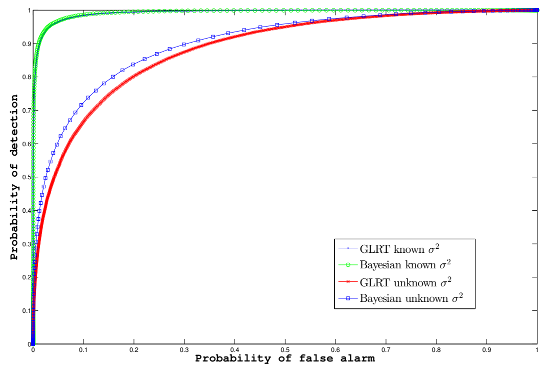

Fig. 1. ROC for GLRT and bayesian detectors for $M = 4$ sensors, rank $K = 2$ and $100,000$ realizations of length $N = 128$. The noise variance $\sigma^2 = \sigma_a^2 = 0.1$ and SNR=-10dB.

A. PARAMETRIZATION AND INVARIANT MEASURE FOR $\mathbf{G}_{K,N}$

The first part of this appendix follows work of Nicolaescu [18] (see also [19]). Points in $\mathbf{G}_{K,N}(\mathbb{C})$ are in one-to-one correspondence with orthogonal projections P of $\mathbb{C}^N$ onto a K-dimensional subspace V with $\mathrm{Tr}(P) = K$. The integral (10) is thus an integral over $\mathbf{G}_{K,N}$. $\mathbf{G}_{K,N}$ is a smooth complex manifold of complex dimension $K(N-K)$. In order to compute the integral (10), local coordinates on $\mathbf{G}_{K,N}$ about P_0 are needed. To this end, denote by L_0 the subspace of $\mathbb{C}^N$ spanned by the first K vectors $\{e_1, \dots, e_K\}$ of the standard basis. Consider the space of all linear maps $Z : L_0 \to L_0^\perp$. Choosing bases for L_0 and $L_0^\perp$, each such map Z is represented as an $(N-K) \times K$ complex matrix and can be associated with the subspace $L_Z \subset \mathbb{C}^N$ defined by

$$L_Z = \{x + Zx : x \in L_0\} \tag{15}$$

These are called graph subspaces due to their association with the graphs of the maps Z; i.e., $\{(x, Zx) : x \in L_0\}$. The orthogonal complement of L_Z is

$$L_Z^\perp = \{-y + Z^\dagger y : y \in L_0^\perp\} \tag{16}$$

In what follows, it will be useful to construct orthonormal bases for L_Z and $L_Z^\perp$ in terms of orthonormal bases for L_0 and $L_0^\perp$. For L_0 and $L_0^\perp$, take the standard basis $\{e_1, \dots, e_N\}$ of $\mathbb{C}^N$ and note that

$$\begin{array}{llll} e_j & \text{for} & j = 1, \dots, K & \text{span} \quad L_0 \\ e_j & \text{for} & j = 1, \dots, N & \text{span} \quad L_0^\perp \end{array}$$

It is straightforward to verify that

$$f_j = (I_0 + Z^\dagger Z)^{-\frac{1}{2}} e_j + Z(I_0 + Z^\dagger Z)^{-\frac{1}{2}} e_j, j = 1, \dots, K \tag{17}$$

is an orthonormal basis for L_Z and

$$f_j = -(I_{0^\perp} + ZZ^\dagger)^{-\frac{1}{2}} e_j + Z^\dagger (I_{0^\perp} + ZZ^\dagger)^{-\frac{1}{2}} e_j, j=K+1, \dots, N \tag{18}$$

Note that both $(I_0 + Z^\dagger Z)$ and $(I_{0^\perp} + ZZ^\dagger)$ are Hermitian and positive definite. The orthogonal projection onto L_Z may now be written as the block matrix

$$P_Z = \begin{pmatrix} (I_K + Z^\dagger Z)^{-1} & (I_K + Z^\dagger Z)^{-1} Z^\dagger \\ Z(I_K + Z^\dagger Z)^{-1} & Z(I_K + Z^\dagger Z)^{-1} Z^\dagger \end{pmatrix} \tag{19}$$

This yields a smoothly parametrized set

$$\Gamma = \{P_Z : Z \in \mathbb{C}^{(N-K)\times K}\} \tag{20}$$

which defines an open subset of $\mathbf{G}_{K,N}$ containing P_0. This open subset covers all of $\mathbf{G}_{K,N}$ except a set of measure zero.

In order to define non-informative priors on $\mathbf{G}_{K,N}$ and to compute marginalization integrals over $\mathbf{G}_{K,N}$ it is necessary to define an invariant measure. This is conveniently achieved through the use of differential forms following James [20]. By analogy with James' invariant measure for the real grassmannian, the invariant volume form (measure) on $\mathbf{G}_{K,N}$ can be defined as

$$w = (-2i)^{-K(N-K)} \left(\bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} \langle h_j, db_i \rangle \right) \wedge \left(\bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} \langle db_i, h_j \rangle \right)$$

In this expression, at each point L in $\mathbf{G}_{K,N}, \{b_i, \; i = 1, \dots, K\}$ is an orthonormal basis for L and $\{h_j; j = 1, \dots, N-K\}$ is an orthonormal basis for $L^\perp$. Using the bases (17) and (18),

$$\begin{aligned}
&\bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} \langle f_j, df_i \rangle \\
&= \bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} -\langle e_j, (I + ZZ^\dagger)^{-\frac{1}{2}} dZ (I + Z^\dagger Z)^{-\frac{1}{2}} e_i \rangle \\
&= (-1)^{K(N-K)} \bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} [(I + ZZ^\dagger)^{-\frac{1}{2}} \otimes (I + Z^\dagger Z)^{-\frac{1}{2}} dZ]_{ji} \\
&= (-1)^{K(N-K)} \det[(I+ZZ^\dagger)^{-\frac{1}{2}} \otimes (I+Z^\dagger Z)^{-\frac{1}{2}}] \bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} dz_{ji} \\
&= (-1)^{K(N-K)} \det(I+ZZ^\dagger)^{-\frac{K}{2}} \det(I+Z^\dagger Z)^{-\frac{(N-K)}{2}} \bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} dz_{ji} \\
&= (-1)^{K(N-K)} \det(I + Z^\dagger Z)^{-\frac{N}{2}} \bigwedge_{i=1}^{K} \bigwedge_{j=1}^{N-K} dz_{ji}
\end{aligned}$$

Thus

$$w = 2^{-K(N-K)} i^{K^2(N-K)^2} \det(I+Z^\dagger Z)^{-N} \bigwedge_{j=1}^{K} \bigwedge_{i=1}^{N-K} dz_{ij} \wedge d\bar{z}_{ij}$$

and so the normalized invariant measure on grassmannian $\mathbf{G}_{K,N}$ is

$$d\mu(Z) = \frac{1}{\mathrm{vol}(G_{K,N})} \det(I + Z^\dagger Z)^{-N} \prod_{i=1}^{N-K} \prod_{j=1}^{K} d\Re(z_{ij}) d\Im(z_{ij})$$

where $\mathrm{vol}(G_{K,N})$ denotes the volume of the grassmanian

$$\begin{aligned}
\mathrm{vol}(G_{K,N}) &= \int_{G_{K,N}} \det(I + Z^\dagger Z)^{-N} \prod_{i=1}^{K} \prod_{j=1}^{N-K} d\Re(z_{ij}) d\Im(z_{ij}) \\
&= \frac{\prod_{\ell=N-K+1}^{N} A_{2\ell-1}}{\prod_{\ell=1}^{K} A_{2\ell-1}}
\end{aligned}$$

where A_ℓ is the area of the unit sphere in $\mathbb{R}^\ell$,

$$A_\ell = \frac{2\pi^{\ell/2}}{\Gamma(\ell/2)}$$

7. REFERENCES

[1] G. C. Carter, "Coherence and time delay estimation," *Proceedings of the IEEE*, vol. 75, no. 2, pp. 236–255, 1987.

[2] G. C. Carter and A. H. Nuttall, "Statistics of the estimate of coherence," *Proceedings of the IEEE*, vol. 60, pp. 465–466, April 1972.

[3] A. H. Nuttall, "Invariance of distribution of coherence estimate to second-channel statistics," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 29, no. 2, pp. 120–122, 1981.

[4] H. Gish and D. Cochran, "Invariance of the magnitude-squared coherence estimate with respect to second-channel statistics," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 35, no. 12, pp. 1774–1776, 1987.

[5] G. C. Carter, *Coherence and Time Delay Estimation*. IEEE Press, 1993.

[6] R. D. Trueblood and D. L. Alspach, "Multiple coherence as a detection statistic," Naval Ocean Systems Center, Tech. Rep. NOSC 265, 1978.

[7] H. Gish and D. Cochran, "Generalized coherence," in *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, April 1988, pp. 2745–2748.

[8] D. Cochran and H. Gish, "Multiple-channel detection using generalized coherence," in *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, April 1990, pp. 2883–2886.

[9] D. Cochran, H. Gish, and D. Sinno, "A geometric approach to multiple-channel signal detection," *IEEE Transactions on Signal Processing*, vol. 43, no. 9, pp. 2049–2057, 1995.

[10] A. Clausen and D. Cochran, "An invariance property of the generalized coherence estimate," *IEEE Transactions on Signal Processing*, vol. 45, no. 4, pp. 1065–1067, 1997.

[11] ——, "Asymptotic analysis of the generalized coherence estimate," *IEEE Transactions on Signal Processing*, vol. 49, no. 1, pp. 45–53, 2001.

[12] D. Ramírez, J. Vía, I. Santamaría, and L. L. Scharf, "Detection of spatially correlated Gaussian time series," *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 5006–5015, 2010.

[13] D. Ramírez, G. Vazquez-Vilar, R. López-Valcarce, J. Vía, and I. Santamaría, "Detection of rank-p signals in cognitive radio networks with uncalibrated multiple antennas," *IEEE Transactions on Signal Processing*, vol. 59, no. 8, pp. 3764–3775, 2011.

[14] D. Ramírez, J. Iscar, J. Vía, I. Santamaría, and L. L. Scharf, "The locally most powerful invariant test for detecting a rank-p gaussian signal in white noise," in *Proceedings of the IEEE Sensor Array and Multichannel Signal Processing Workshop*, June 2012.

[15] S. Sirianunpiboon, S. D. Howard, and D. Cochran, "A Bayesian derivation of generalized coherence detectors," in *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, March 2012, pp. 3253–3256.

[16] A. Horn, "Doubly stochastic matrices and the diagonal of a rotation matrix," *American Journal of Mathematics*, vol. 76, pp. 620–630, 1954.

[17] A. Gelman, J. B. Carlin, H. S. Stern, and D. B. Rubin, *Bayesian Data Analysis*. CRC Press, 2003.

[18] L. Nicolaescu, *An Invitation to Morse Theory*. Universitext, Springer, 2007.

[19] M. C. Hansen, "Morse theory on complex grassmannians," http://www.math.ku.dk, 2012.

[20] A. T. James, "Normal multivariate analysis and the orthogonal group," *Annals of Mathematical Statistics*, vol. 25, no. 1, pp. 40–75, 1954.