# Chapter 1
# Positive Operator Valued Measures: A General Setting for Frames

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**Abstract**  This paper presents an overview of close parallels that exist between the theory of positive operator-valued measures (POVMs) associated with a separable Hilbert space and the theory of frames on that space, including its most important generalizations. The concept of a framed POVM is introduced, and classical frames, fusion frames, generalized frames, and other variants of frames are all shown to to arise as framed POVMs. This observation allows drawing on a rich existing theory of POVMs to provide new perspectives in the study of frames.

## 1.1 Introduction

Frames have become a standard tool in signal processing, allowing uniform description of many linear but non-orthogonal transform techniques that underpin a wide variety of signal and image processing algorithms. Initially popularized in connection with wavelet applications, frames are now a standard tool in sampling, compression, array processing, as well as in spectral and other transform methods for time series.

Frames were initially introduced in a 1952 paper of Duffin and Schaeffer [10], where they appeared as an abstraction of sampled Fourier transforms. Little interest was shown in them until the appearance of the 1986 paper

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[8] by Daubechies, Grossmann, and Meyer which coincided with the rise of wavelet methods in signal processing. Subsequently they were taken up by numerous authors. Several excellent sources, including [7, 14, 11, 15], are available for further details of both the theory and the many applications of frames.

The standard definition of a frame is as a collection $\mathcal{F} = \{\varphi_k : k \in K\}$ of elements of a separable Hilbert space $\mathcal{H}$. The index set $K$ may be finite or infinite. In order for $\mathcal{F}$ to constitute a frame, there must exist constants $0 < A \leq B < \infty$ such that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{k \in K} |\langle \varphi_k, f \rangle|^2 \leq B\|f\|^2. \tag{1.1}$$

Roughly speaking, a projection $f \mapsto \langle f, \varphi_k \rangle$ of a vector $f$ representing the state of a system onto an individual element $\varphi_k$ of a frame may be seen as a measurement of that system, and the aim is to reconstruct the state $f$ from the collection of all individual measurements $\{\langle f, \varphi_k \rangle : k \in K\}$ in a robust way. The frame condition as stated in (1.1) expresses the ability to do that, and the frame bounds $A$ and $B$ provide a measure of robustness. If $A = B$, the frame is said to be *tight*. Orthonormal bases are special cases of tight frames, and for these $A = B = 1$.

Several generalizations of the basic concept of a frame have been proposed. These include, in particular, the possibility that the family $\{\varphi_k : k \in K\} \subset \mathcal{H}$ is indexed by a continuum rather than a discrete index set, resulting in what are called *generalized frames*. There are various formulations of generalized frames in the literature; see in particular [1]. From the perspective of this paper, the infrastructure of a generalized frame is a measurable function from a measure space, which serves the role of the index set, to $\mathcal{H}$. Specifically, let $(\Omega, \mathcal{B}, \mu)$ be a measure space (e.g., $\Omega = \mathbb{R}$ with $\mathcal{B}$ its Borel sets and $\mu$ Lebesgue measure) and let $\Phi : \Omega \to \mathcal{H}$ be a $\mu$-measurable function. The collection $\{\Phi(t) : t \in \Omega\} \subset \mathcal{H}$ is a generalized frame for $\mathcal{H}$ if it satisfies a condition analogous to the frame condition (1.1); i.e., for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \int_{\Omega} |\langle \Phi(t), f \rangle|^2 \, d\mu(t) \leq B\|f\|^2. \tag{1.2}$$

Define $\Pi_{\varphi} : \mathcal{H} \to \mathcal{H}$ to be orthogonal projection into the one-dimensional subspace spanned by the unit-norm element $\varphi \in \mathcal{H}$; i.e., $\Pi_{\varphi}(f) = \langle \varphi, f \rangle \varphi$. With this notation, (1.2) becomes

$$A\mathbb{I} \leq \int_{\Omega} \Pi_{\Phi(t)} \, d\mu(t) \leq B\mathbb{I}, \tag{1.3}$$

where $\mathbb{I}$ denotes the identity operator on $\mathcal{H}$ and the inequalities mean that the differences are positive definite operators on $\mathcal{H}$. The integral in (1.3) is in the weak sense; i.e., for a suitable measurable family of operators $\{S(t) : t \in \Omega\}$

on $\mathcal{H}$, the integral $\int_{\Omega} S(t) \, d\mu(t)$ is defined to be the operator $D$ satisfying

$$\langle f, D\varphi \rangle = \int_{\Omega} \langle f, S(t)\varphi \rangle \, d\mu(t)$$

for $f$ and $\varphi$ in $\mathcal{H}$.

*Fusion frames* generalize the concept of a frame in a different direction. They have received considerable recent attention in the signal processing literature; see, for example, [12, 5, 20, 3]. In a fusion frame, the one-dimensional projections $\Pi_{\varphi_k}$ are replaced by projections $\Pi_k$ onto potentially higher dimensional closed subspaces $W_k \subset \mathcal{H}$. Thus a fusion frame $\mathcal{F}$ is a family $\{(W_k, w_k) : k \in K\}$ of closed subspaces of $\mathcal{H}$ and a corresponding family of weights $w_k \geq 0$ satisfying the frame condition

$$A\|f\|^2 \leq \sum_{k \in K} w_k^2 \|\Pi_k(f)\|^2 \leq B\|f\|^2 \tag{1.4}$$

for all $f \in \mathcal{H}$. Some authors have promoted fusion frames as a means of representing the problem of fusion of multiple measurements in, for example, a sensor network. In this view, each projection corresponds to a node of the network, and the fusion frame itself, as its name suggests, provides the mechanism for fusion of these measurements centrally.

Not surprisingly, the ideas of generalized frames and fusion frames can be combined into a composite generalization. A *generalized fusion frame* $\mathcal{F}$ for $\mathcal{H}$ consists of a pair of measurable functions $(\Phi, w)$. In this setting, $w : \Omega \to \overline{\mathbb{R}_+}$ and $\Phi : \Omega \to \mathcal{P}(\mathcal{H})$ where $\mathcal{P}(\mathcal{H})$ denotes the space of orthogonal projections of any rank (including possibly $\infty$) on $\mathcal{H}$, endowed with the weak operator topology. Measurability of $\Phi$ is in the weak sense that $t \mapsto \langle \varphi, \Phi(t)\psi \rangle$ is $\mu$-measurable for each $\varphi$ and $\psi$ in $\mathcal{H}$. As part of the definition, it is also required that the function $t \mapsto \Phi(t)f$ is in $L_2(\Omega, \mu)$ for each $f \in \mathcal{H}$. The frame condition in operator form, as in (1.3), becomes

$$A\mathbb{I} \leq \int_{\Omega} w(t)^2 \Phi(t) \, d\mu(t) \leq B\mathbb{I}.$$

As described in later sections of this paper, this definition of a generalized fusion frame leads to a concept that is, in effect if not in formalism, remarkably similar to that of a positive operator-valued measure (POVM) — a concept that has been prevalent in the quantum physics literature for many years. This is hardly unexpected from a signal processing viewpoint, as the concept of POVM was introduced and developed in quantum mechanics as a means to represent the most general form of quantum measurement of a system. Further, connections between POVMs and frames have been noted frequently in the physics literature (e.g., [4, 18]), although these relationships seem to be unmentioned in mathematical work on frames.

The remainder of the paper develops a generalization of the POVM concept as used in quantum mechanics, which encompasses the theory of frames — including all of the generalizations discussed above. Once generalized fusion frames are accepted, setting the discourse in terms of POVMs enables the importation of much theory from the quantum mechanics literature and also brings to light some decompositions that are not readily apparent from the frame formalism.

A key result used in what follows is the classical theorem of Naimark [16] which, long before frames became popular in signal processing or POVMs were used in quantum mechanics, formalized analysis and synthesis in this general context. When applied to the cases above, Naimark's perspective exactly reproduces those notions.

Subsequent sections describe positive operator valued measures, introduce the theorem of Naimark, and discuss how POVMs relate to frames and their generalizations. In this brief description of the relationship between POVMs and the generalizations of frames, it will only be possible to touch on the power of the POVM formalism.

## 1.2 Analysis and Synthesis

The various concepts of frame, fusion frame, and generalized frame all give rise to analysis and synthesis operations. In the case of a frame, a prevalent point of view is that an analysis operator $F$ takes a "signal" in $\mathcal{H}$ to a set of complex "coefficients" in the space $\ell_2(K)$ of square-summable sequences on the index set $K$; i.e., $F$ is the Bessel map given by $F(f) = \{\langle f, \varphi_k \rangle : k \in K\}$ where the finiteness of the upper frame bound $B$ guarantees the square-summability of this coefficient sequence. The synthesis operator is the adjoint map $F^* : \ell_2(K) \to \mathcal{H}$, given by

$$F^*(\{a_k\}) = \sum_{k \in K} a_k \varphi_k,$$

and corresponds to synthesis of a signal from a set of coefficients. It follows directly from (1.1) that the *frame operator* $\mathbb{F} = F^* F$ satisfies

$$A\mathbb{I} \leq \mathbb{F} \leq B\mathbb{I}. \tag{1.5}$$

To accommodate developments later in this paper, it is useful to describe analysis and synthesis with frames in a slightly different way. With each $\varphi_k$ in the frame $\mathcal{F}$, associate the one-dimensional orthogonal projection operator $\Pi_k$ that takes $f \in \mathcal{H}$ to

$$\Pi_k(f) = \frac{\langle \varphi_k, f \rangle}{||\varphi_k||^2} \, \varphi_k$$

Note that $\Pi_k : \mathcal{H} \to W_k$ where $W_k$ is the one-dimensional subspace of $\mathcal{H}$ spanned by $\varphi_k$. Also, $\|\Pi_k(f)\| = |\langle \varphi_k, f \rangle| / \|\varphi_k\|$. Thus the frame condition (1.1) is equivalent to

$$A\|f\|^2 \leq \sum_{k \in K} w_k^2 \|\Pi_k(f)\|^2 \leq B\|f\|^2$$

where $w_k = |\langle \varphi_k, f \rangle| \geq 0$. From a comparison of this expression with (1.4), it is clear that the weights $w_k$ account for the possibility that the frame elements $\varphi_k \in \mathcal{F}$ are not of unit norm. Although it is typical to think of the analysis operator as producing a set of coefficients for each signal $f \in \mathcal{H}$ via the Bessel map, as described above, it is more suitable for generalization to regard it as a map from $\mathcal{H}$ to $\mathcal{H}$ that "channelizes" $f$ into signals $w_k \Pi_k(f) \in W_k \subset \mathcal{H}$. The synthesis operator is then a linear rule for combining a set of signals from the channels $W_k$ to form an aggregate signal in $\mathcal{H}$.

With this view, the analysis operator for a fusion frame is a natural generalization of its frame counterpart in which the subspaces $W_k$ can be of dimension greater than one and the projection operators $\Pi_k$ are from $\mathcal{H}$ to $W_k$. The analysis operator is $F : \mathcal{H} \to \bigoplus_{k \in K} W_k$ given by

$$F(f) = \{w_k \Pi_k(f) : k \in K\} \in \bigoplus_{k \in K} W_k.$$

The adjoint map $F^* : \bigoplus_{k \in K} W_k \to \mathcal{H}$ is given by

$$F^*(\{\xi_k\}) = \sum_{k \in K} w_k \xi_k \in \mathcal{H}, \quad \{\xi_k : k \in K\} \in \bigoplus_{k \in K} W_k.$$

The frame bound conditions guarantee that everything is well-defined. The corresponding fusion frame operator $\mathbb{F} = F^*F : \mathcal{H} \to \mathcal{H}$ is given by

$$\mathbb{F}(f) = \sum_{k \in K} w_k^2 \Pi_k(f),$$

and the same kind of frame bound inequality as in (1.5) holds for fusion frames.

For the generalized frame described in Section 1.1, the analysis operator $F : \mathcal{H} \to L_2(\Omega, \mu)$ is given by

$$F(f)(t) = \langle f, \Phi(t) \rangle, \quad t \in \Omega, \ f \in \mathcal{H},$$

and its adjoint by

$$F^*(u) = \int_\Omega u(t) \Phi(t) \, d\mu(t) \in \mathcal{H}, \quad u \in L^2(\Omega, \mu).$$

Again, the generalized frame operator $\mathbb{F} = F^*F$ satisfies inequalities (1.5).

For generalized fusion frames there is a corresponding definition of analysis and synthesis operators, but its description requires the definition of direct integrals of Hilbert spaces [9]. In any case the ideas will be subsumed under the more general development to follow.

It is immediately evident that, in each case discussed above, the synthesis operator does not reconstruct the analyzed signal; i.e., in general $F^*F \neq \mathbb{I}$. In the case of a frame, inversion of the analysis operator is performed by invoking a *dual frame*. There are various different usages of this terminology in the literature (see [5, 13, 15]). For the purposes here, given a frame $\{\varphi_k\}$ for the Hilbert space $\mathcal{H}$, a dual frame $\{\tilde{\varphi}_k\}$ satisfies

$$f = \sum_{k \in K} \langle \varphi_k, f \rangle \tilde{\varphi}_k = \sum_{k \in K} \langle \tilde{\varphi}_k, f \rangle \varphi_k. \tag{1.6}$$

In other words, the dual frame inverts the analysis and synthesis operations of the original frame to give perfect reconstruction. Such a dual frame always exists; indeed, it is easy to verify that

$$\tilde{\varphi}_k = \mathbb{F}^{-1}(\varphi_k) \tag{1.7}$$

has the appropriate property. Dual frames as defined in (1.6) are not in general unique; the one in equation (1.7) is called the *canonical dual frame*. In the case of a fusion frame $\{(W_k, w_k) : k \in K\}$, there also exist dual fusion frames. The canonical dual fusion frame is $\{(\mathbb{F}^{-1}W_k, w_k) : k \in K\}$. See [13] for proofs of the existence and discussion of the properties of dual frames in this context.

## 1.3 Positive Operator Valued Measures

The goal of this section is to define a *framed POVM* and give some examples of such objects. Consider a topological space $\Omega$ which, to avoid technicalities, will be assumed to be "nice"; e.g., a complete separable metric space or a locally compact second countable space. The crucial point is that $\Omega$ has sufficient structure to make the concept of regularity of measures meaningful and useful, though regularity will not be explicitly discussed in this paper. Denote by $\mathcal{B}(\Omega)$ the $\sigma$-algebra of Borel sets on $\Omega$ and by $\mathcal{P}(\mathcal{H})$ the space of positive operators on a Hilbert space $\mathcal{H}$. A *framed POVM* is a function $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ satisfying the following two conditions:

POVM-1) For all $f$ in $\mathcal{H}$, $\omega \mapsto \langle f, M(\omega)f \rangle$ is a regular Borel measure on $\mathcal{B}(\Omega)$, denoted by $\mu_f$, and

POVM-2) $A\mathbb{I} \leq M(\Omega) \leq B\mathbb{I}$ for some $0 < A \leq B < \infty$.

As in the case of frames, the numbers $A$ and $B$ are called the frame bounds for $M$. Without the condition POVM-2), the object is called a POVM; i.e.,

without the epithet "framed." Such a function is a measure on $\mathcal{B}(\Omega)$ that takes values in the set of positive operators on $\mathcal{H}$, though the countable aspect of its additivity is only in a weak sense. In the quantum mechanics context, POVM-2) is replaced by the more strict requirement that $M(\Omega) = \mathbb{I}$. A framed POVM is *tight* if $A = B$, and if $A = B = 1$, $M$ is a *probability POVM*. Probability POVMs are used in quantum mechanics as the most general form of quantum measurement.

As an example of a framed POVM, consider a fusion frame $\{(W_k, w_k) : k \in K\}$ in $\mathcal{H}$. Define $\Omega = K$ with the $\sigma$-field $\mathcal{B}(\Omega)$ taken to be the power set of $\Omega$. Denoting, as above, projection onto $W_k$ by $\Pi_k$,

$$M(\omega) = \sum_{k \in \omega} w_k \Pi_k. \tag{1.8}$$

It is straightforward to see that this satisfies both parts of the definition of a framed POVM, with the frame bounds being the bounds in the definition of the fusion frame. Thus every fusion frame, and hence every frame, is trivially represented as a framed POVM.

If $\mathcal{F} = \{\Phi(t) : t \in \Omega\}$ is a generalized frame for $\mathcal{H}$, a POVM $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ can be defined by

$$M(\omega) = \int_\omega \Pi_{\Phi(t)} \, d\mu(t), \tag{1.9}$$

where $\Pi_{\Phi(t)}$ denotes projection into the one-dimensional subspace of $\mathcal{H}$ spanned by $\Phi(t)$. $M$ is a framed POVM with the same frame bounds as those of $\mathcal{F}$.

As will be discussed in Section 1.5, POVMs provide a rather general framework for analysis and reconstruction of signals. It will be seen that framed POVMs are only slightly more general than generalized fusion frames discussed briefly in Section 1.1. The impetus for studying POVMs in this context arises in part from the opportunity to draw on existing theory about POVMs in the physics literature for development and description of new constructs in signal processing. Some examples in this paper illustrate this possibility, though much of the formalism is left for a later paper.

## 1.4 Spectral Measures and the Naimark Theorem

A POVM $S$ is a *spectral* POVM if

$$S(\omega_1 \cap \omega_2) = S(\omega_1)S(\omega_2), \qquad \omega_1, \omega_2 \in \mathcal{B}(\Omega).$$

Spectral POVMs arise, for example, in the spectral theorem for Hermitian operators on Hilbert space (see for example [21]). If $S$ is a spectral POVM,

then $S(\Omega)$ is a projection, and every $S(\omega)$ with $\omega \in \mathcal{B}(\Omega)$ is a projection dominated by $S(\Omega)$; i.e.,

$$S(\omega)S(\Omega) = S(\Omega)S(\omega) = S(\omega).$$

Thus, for any $\omega \in \mathcal{B}(\Omega)$, $S(\omega)$ is completely specified by its behavior on the closed subspace $S(\Omega)\mathcal{H}$ of $\mathcal{H}$. Consequently, for most purposes it suffices to assume $S(\Omega) = \mathbb{I}_{\mathcal{H}}$. In particular, if a spectral POVM is framed, then this condition must hold; conversely, imposing this condition on a spectral POVM ensures that it is framed. Since the interest here is on framed POVMs, it will be assumed that $S(\Omega) = \mathbb{I}_{\mathcal{H}}$ whenever a spectral POVM appears in subsequent discussion in this paper. Note that, while a spectral POVM $S$ need not be probability POVM in general, the condition that it is framed implies that $S$ will be a probability POVM. Intuitively, spectral POVMs play an analogous role relative to framed POVMs to the one played by orthogonal bases relative to frames; i.e., spectral POVMs generalize orthogonal bases in a sense similar to that in which framed POVMs generalize frames.

With this machinery in place, it is possible to state the key theorem on POVMs due to Naimark [16], who formulated the result for POVMs without the framed condition. The following version is a relatively straightforward adaptation to framed POVMs.

**Theorem 1.** *Suppose $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ is a framed POVM with frame bounds $A$ and $B$. Then there is an "auxiliary" Hilbert space $\mathcal{H}^{\sharp}$, a spectral POVM $S$ with values in $\mathcal{P}(\mathcal{H}^{\sharp})$, and a bounded linear map $V : \mathcal{H}^{\sharp} \to \mathcal{H}$ such that*

$$M(\omega) = VS(\omega)V^*, \qquad \omega \in \mathcal{B}(\Omega)$$

*and $A\mathbb{I} \le VV^* \le B\mathbb{I}$.*

For developments later in this chapter, it will be useful to have a sketch of the proof of this theorem. Given a POVM $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$, consider the linear space $\mathcal{L}$ of $\mathcal{H}$-valued simple functions on $\Omega$; i.e., finite linear combinations of functions of the form

$$\xi_{\omega}(t) = \begin{cases} \xi & \text{if } t \in \omega \\ 0 & \text{otherwise,} \end{cases} \tag{1.10}$$

where $\omega \in \mathcal{B}(\Omega)$ and $\xi \in \mathcal{H}$. A pre-inner product on $\mathcal{L}$ is obtained by defining

$$\langle \xi_{\omega}, \xi'_{\omega'} \rangle_{\mathcal{L}} = \langle M(\omega)\xi, M(\omega')\xi' \rangle_{\mathcal{H}}. \tag{1.11}$$

Completion followed by factoring out zero-length vectors produces $\mathcal{H}^{\sharp}$, as a Hilbert space. The map from $\mathcal{H}$ to $\mathcal{L}$ taking $\xi$ to $\xi_{\Omega}$ results in $V^* : \mathcal{H} \to \mathcal{H}^{\sharp}$ and $V$ takes $\xi_{\omega}$ to $M(\Omega)^*M(\omega)\xi$. The spectral measure $S$ arises first on $\mathcal{L}$ as

$$S(\omega')(\xi_{\omega}) = \xi_{\omega \cap \omega'} \qquad \xi \in \mathcal{H}, \quad \omega, \omega' \in \mathcal{B}(\Omega), \tag{1.12}$$

and then carries over to $\mathcal{H}^\sharp$.

The collection $(S, \mathcal{H}^\sharp, V)$ is known as a *Naimark representation* of the framed POVM $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$. Further, a Naimark representation is *minimal* if the set

$$\{S(\omega)V^*\varphi : \varphi \in \mathcal{H},\ \omega \in \mathcal{B}(\Omega)\}$$

is dense in $\mathcal{H}^\sharp$. Minimal Naimark representations are essentially unique in the sense that if $(S, \mathcal{H}_\sharp, V)$ and $(S', \mathcal{H}'_\sharp, V')$ are two such representations for the same $M$, then there is a surjective isometry $T : \mathcal{H}_\sharp \to \mathcal{H}'_\sharp$ such that $V'T = V$ and $T^{-1}S'(\omega)T = S(\omega)$ for all $\omega \in \mathcal{B}(\Omega)$. A fashionable way to handle the Naimark representation in recent literature (see [17]) is to convert POVMs to (completely) positive operators on commutative $C^*$-algebras via integration. In this setting, Naimark's theorem becomes a special case of Stinespring's theorem [19], which does not require commutativity of the $C^*$-algebra. A full description of this approach would be tangential to this paper.

*Example 1. Consider a generalized frame $\Phi : \Omega \to \mathcal{H}$ on the measure space $(\Omega, \mu)$ with frame bounds $A \le B$. $\Phi$ gives rise to a framed POVM $M$ as in (1.9). To form a Naimark representation for $M$, define the Hilbert space $\mathcal{H}^\sharp$ to be $L_2(\Omega, \mu)$ and let the spectral measure $S$ be the canonical one on this space; i.e.,*

$$S(\omega)f(t) = \mathbb{1}_\omega(t)f(t), \qquad f \in L_2(\Omega, \mu).$$

*$S$ is clearly a spectral measure since the characteristic functions satisfy $\mathbb{1}_\omega \mathbb{1}_{\omega'} = \mathbb{1}_{\omega \cap \omega'}$. The map $V : L_2(\Omega, \mu) \to \mathcal{H}$ is defined by*

$$V(f) = \int_\Omega f(t)\Phi(t)\,d\mu(t)$$

*where $f(t)\Phi(t)$ is the product of the scalar $f(t)$ and $\Phi(t) \in \mathcal{H}$. It can be verified that this is indeed a (the) minimal Naimark representation of $M$.*

*Example 2. Let $\mathcal{F} = \{(W_k, w_k) : k \in K\}$ be a fusion frame in $\mathcal{H}$. $\mathcal{F}$ corresponds to a framed POVM as in (1.8). In this case, $\mathcal{H}^\sharp$ may be taken to be the formal direct sum $\bigoplus_{k \in K} W_k$. The appropriate spectral measure $S$ is defined on subsets $J$ of $\Omega = K$ by*

$$S(J) = \bigoplus_{k \in J} \Pi_k \tag{1.13}$$

*where $\Pi_k$ is the projection into $W_k$ in $\mathcal{H}^\sharp$. Writing an element $f$ of $\mathcal{H}^\sharp$ as $f = \{f_k \in W_k : k \in K\}$, the map $V : \mathcal{H}^\sharp \to \mathcal{H}$ is given by*

$$V(f) = \sum_{k \in K} w_k f_k, \tag{1.14}$$

*where the terms in the sum are considered as elements of $\mathcal{H}$. The square-summability of the weights $w_k$ guarantees that the sum in (1.14) converges in $\mathcal{H}$ because the Cauchy-Schwarz inequality gives*

$$\sum_{k \in K} \|w_k \varphi_k\| \leq \Big(\sum_{k \in K} w_k^2\Big)^{1/2} \Big(\sum_{k \in K} \|\varphi_k\|^2\Big)^{1/2}. \qquad (1.15)$$

*Thus $V : \mathcal{H}^\sharp \to \mathcal{H}$ is a bounded linear map; in fact, by (1.15),*

$$\|V\| \leq \Big(\sum_{k \in K} w_k^2\Big)^{1/2}.$$

*Its adjoint $V^* : \mathcal{H} \to \mathcal{H}^\sharp$ is given by*

$$V^*(\varphi) = \{w_k \Pi_k(\varphi) : k \in K\}.$$

*Setting $\omega = \Omega = K$ gives $S(\Omega) = \mathbb{I}$ and*

$$M(\Omega) = V S(\Omega) V^* = V V^*,$$

*The frame bounds imply $A \leq V V^* \leq B$ and, if the fusion frame is tight, then $V V^* = A\mathbb{I}$.*

From a comparison of the descriptions in Section 1.2 with the examples given here, it is evident that Naimark's Theorem provides exactly the machinery for discussing analysis and synthesis operators in a general context. This is undertaken in the next section.

## 1.5 Analysis and Synthesis for General POVMs

The preceding examples indicate that the Naimark representation provides a mechanism for analysis and synthesis in POVMs that precisely extends the corresponding ideas for frames and fusion frames. To be specific, let $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ be a POVM and let $(S, \mathcal{H}^\sharp, V)$ be the corresponding mimimal Naimark representation. In this context, $\mathcal{H}^\sharp$ will be called the *analysis space* and $V^* : \mathcal{H} \to \mathcal{H}^\sharp$ the *analysis operator*. Similarly, $V : \mathcal{H}^\sharp \to \mathcal{H}$ will be called the *synthesis operator*. The use of this terminology is directly analogous to the way it is used for frames and their generalizations in Section 1.2. Further, the Naimark representation also provides a means, via the spectral measure $S$, for keeping track of the labeling of the POVM.

Analysis of an element $f \in \mathcal{H}$ is the $\mathcal{H}^\sharp$-valued measure $\mathcal{A}$ on $\mathcal{B}(\Omega)$ defined by

$$\mathcal{A}(f)(\omega) = \hat{f}(\omega) = S(\omega) V^* f \in \mathcal{H}^\sharp. \qquad (1.16)$$

In the case of a frame $\{\varphi_k : k \in K\}$, this measure on subsets of $\Omega = K$ associates the "coefficients" $\langle f, \varphi_k \rangle e_k \in \ell_2(K)$ with the signal $f$, where $\{e_k : k \in K\}$ is the standard basis of $\ell_2(K)$. Given a measure $\rho : \mathcal{B}(\Omega) \to \mathcal{H}^\sharp$ as in (1.16), the synthesis operator takes $\rho$ to

$$\mathcal{S}(\rho) = V \int_{\Omega} d\rho(t) \in \mathcal{H}. \tag{1.17}$$

As the examples in the preceding sections show, these analysis and syntheses operators correspond precisely to those of classical frames, fusion frames, and generalized fusion frames.

## 1.6 Isomorphism of POVMs

Two POVMs $(M_1, \Omega, \mathcal{H}_1)$ and $(M_2, \Omega, \mathcal{H}_2)$ are isomorphic if there is a surjective unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U M_1(\omega) U^{-1} = M_2(\omega)$ for all $\omega \in \mathcal{B}(\Omega)$. The following result is a straightforward consequence of the proof of the Naimark theorem.

**Theorem 2.** *Suppose that POVMs $M_1 : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H}_1)$ and $M_2 : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H}_2)$ are isomorphic via the unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$. Let $(S_1, \mathcal{H}_1^\sharp, V_1)$ and $(S_2, \mathcal{H}_2^\sharp, V_2)$ be minimal Naimark representations of $M_1$ and $M_2$, respectively. Then there is a unitary transformation $U^\sharp : \mathcal{H}_1^\sharp \to \mathcal{H}_2^\sharp$ such that $U^\sharp S_1(\omega)(U^\sharp)^{-1} = S_2(\omega)$ for all $\omega \in \mathcal{B}(\Omega)$ and the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{H}_1 & \xrightarrow{U} & \mathcal{H}_2 \\
V_1 \uparrow & & \uparrow V_2 \\
\mathcal{H}_1^\sharp & \xrightarrow{U^\sharp} & \mathcal{H}_2^\sharp
\end{array}
.
$$

Although this result does not appear to be explicitly stated in the literature, it is implicit in many applications of the Naimark and Stinespring theorems. In particular, the paper of Arveson [2] discusses related ideas. The proof follows by consideration of the construction of the Naimark representation using Hilbert space valued functions as described in Section 1.4. Specifically, using the notation of the sketch proof of Naimark's theorem given in Section 1.4, observe that for the isomorphic POVMS $M_1$ and $M_2$, $U$ gives rise to a map $\mathcal{L}_1 \to \mathcal{L}_2$ taking $\xi_\omega$ to $U(\xi)\omega$ which then produces $U^\sharp$. Moreover, it follows from the definition of the spectral measure in (1.12) that $U^\sharp S_1(\omega)(U^\sharp)^{-1} = S_2(\omega)$ for all $\omega \in \mathcal{B}(\Omega)$.

## 1.7 Canonical Representations and POVMs

Combining the Naimark theorem and Theorem 2 with the canonical representation of spectral POVMs (described in, e.g., [21]) yields a canonical representation of POVMs such that two isomorphic POVMs have the same

canonical representation. This serves to illustrate the utility of the POVM formalism. The canonical representation decomposes $\mathcal{H}^\sharp$, the analysis space of a POVM $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ that arises in its Naimark representation, into a direct sum $\bigoplus_{n \in \mathbb{N}} \mathcal{G}_n$ such that:

1. Each of the spaces $\mathcal{G}_n$ is invariant under the spectral measure; i.e.,

$$S(\omega)\mathcal{G}_n \subset \mathcal{G}_n \qquad \omega \in \mathcal{B}(\Omega), n \in \mathbb{N},$$

   and

2. The restriction of $S$ to $\mathcal{G}_n$ has uniform multiplicity; i.e., $\mathcal{G}_n \simeq \mathbb{C}^{u_n} \otimes L_2(\mu_n)$ if $\mathcal{G}_n$ has finite dimension $u_n$, and $\mathcal{G}_n \simeq \ell_2(\mathbb{N}) \otimes L_2(\mu_n)$ if $\mathcal{G}_n$ is infinite-dimensional.

This representation is essentially unique up to unitary equivalence and replacement of each of the measures $\mu_n$ by one having the same null sets. Denote by $P_n$ the projection into $\mathcal{G}_n$, regarded as a subspace of $\mathcal{H}^\sharp$. Under the (minimal) Naimark representation, $V : \mathcal{H}^\sharp \to \mathcal{H}$ is such that $V^*S(\omega)V = M(\omega)$ for $\omega \in \mathcal{B}(\Omega)$. $V$ can be decomposed as $V = \sum_n VP_n = \sum_n V_n$. The image of $V_n^*$ is in $\mathcal{G}_n$, so that

$$M(\omega) = \sum_n V_n S(\omega) V_n^*, \qquad \omega \in \mathcal{B}(\Omega).$$

The map $M_n : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{G}_n)$, defined by $M_n(\omega) = V_n S(\omega) V_n^*$, is a POVM; more precisely, the values of $M_n$ are positive operators on the closure of the image of $V_n$. The individual measures $M_n$ are themselves POVMs, though they need not be framed even if $M$ is framed. However, $M_n(\Omega) = V_n V_n^*$. Thus if $M$ is a framed POVM with frame bounds $A \leq B$,

$$A\mathbb{I}_\mathcal{H} \leq \sum_n M_n(\Omega) = \sum_n V_n V_n^* \leq B\mathbb{I}_\mathcal{H}.$$

Observe that $V_n^* V_n V_m^* V_m = 0$ for $n \neq m$, since the image of $V_m^*$ lies in $\mathcal{G}_m$ which is in the kernel of $V_n$. So, in an obvious sense,

$$M = \sum_{n \in \mathbb{N}} M_n.$$

Thus every framed POVM is a sum of "uniform multiplicity" POVMs, though these need not be framed, and this composition is essentially unique. The canonical representation is characterized by the sequence of equivalence classes of measures $\{[\mu_n] | n \in \mathbb{N}\}$ and the linear map $V$.

*Example 3. Consider a frame $\mathcal{F} = \{\varphi_k : k \in K\}$ in $\mathcal{H}$ with frame bounds $A \leq B$ and its corresponding framed POVM $M$. In this case $\mathcal{H}^\sharp$ is $\ell_2(K)$, $V : \mathcal{H}^\sharp \to \mathcal{H}$ is given by $V(e_k) = \varphi_k$. The spectral measure on the subsets of $K$ is given by*

$$S(J) = \sum_{k \in J} \Pi_k, \qquad J \subset K,$$

where $\Pi_k$ denotes projection into the subspace of $\ell_2(K)$ spanned by the standard basis element $e_k$. Alternatively, this can be redefined by regarding members of $\ell_2(K)$ as complex-valued functions on $\Omega = K$ and taking $S(J)(f) = \mathbb{1}_J f$ so that the spectral measure is uniform with multiplicity one.

*Example 4. The case of a fusion frame $\{(W_k, w_k) : k \in K\}$ is more complicated than the frame case. The spectral measure $S$ on subsets of $\Omega = K$ is given by (1.13). For each $j \in K$, denote*

$$U_j = \{k \in K : \dim W_k = j\}.$$

*Then*

$$Y_j = \bigoplus_{k \in U_k} W_k \subset \mathcal{H}^\sharp$$

*is isomorphic to $\mathbb{C}^j \otimes \ell_2(U_k)$ or, if $j = \infty$, $\ell_2(U_j) \otimes \ell_2(U_j)$. Evidently, $Y_j$ has uniform multiplicity $j$, and the measure $\mu_j$ is counting measure on $U_j$, provided $U_j$ is not empty. If all $W_k$ for $k \in K$ have the same dimension, then the spectral measure $S$ has uniform multiplicity.*

## 1.8 Dual POVMs

As observed in Section 1.2, each of frame generalizations associates a "dual" object with the frame, and there is a canonical dual in each case. This is also possible for framed POVMs, and indeed is relatively straightforward using the Naimark representation. Consider a POVM $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ and its minimal Naimark representation $(\Omega, S, \mathcal{H}^\sharp, V)$. The *canonical dual POVM* to $M$ is the POVM $\widetilde{M} : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ having Naimark representation $(\Omega, S, \mathcal{H}^\sharp, (VV^*)^{-1}V)$; i.e.,

$$\widetilde{M}(\omega) = (VV^*)^{-1} V S(\omega) V^* (VV^*)^{-1}.$$

The frame condition on $M$ guarantees $0 < A \leq V^*V \leq B < \infty$, which not only ensures the existence of $(VV^*)^{-1}$, but implies $\widetilde{M}$ is a framed POVM with bounds $B^{-1} \leq A^{-1}$ (see Theorem 1). Further,

$$M(\omega)\widetilde{M}(\omega) = \big(VS(\omega)V^*\big)\big((VV^*)^{-1}VS(\omega)V^*(VV^*)^{-1}\big)$$
$$\widetilde{M}(\omega)M(\omega) = \big((VV^*)^{-1}VS(\omega)V^*(VV^*)^{-1}\big)\big(VS(\omega)V^*\big)$$

In particular, invoking the assumption $S(\Omega) = \mathbb{I}_{\mathcal{H}^\sharp}$ gives

$$M(\Omega)\widetilde{M}(\Omega) = \widetilde{M}(\Omega)M(\Omega) = \mathbb{I}_{\mathcal{H}}.$$

From the point of view of analysis and synthesis, if $f \in \mathcal{H}$, its analysis with respect to $M$ is the measure $\mathcal{A}(f)$ given in (1.16). Subsequently applying the synthesis operator $\widetilde{\mathcal{S}}$ associated with the canonical dual POVM $\widetilde{M}$ yields (1.17) gives

$$\widetilde{\mathcal{S}}\mathcal{A}(f)(\Omega) = (VV^*)^{-1}VS(\Omega)V^*f = f.$$

Similarly, analysis of $f$ by $\widetilde{M}$ followed by synthesis with $M$ is also the identity; i.e.,

$$\mathcal{S}\widetilde{\mathcal{A}}(f)(\Omega) = VS(\Omega)V^*(VV^*)^{-1}f = f.$$

## 1.9 Radon-Nikodym Theorem for POVMs

This section summarizes some results pertinent to framed POVMs on finite-dimensional Hilbert spaces. This setting is prevalent in signal processing applications, and it will be seen that the theory developed is valid in a number of infinite-dimensional examples as well. In this setting, the concept of a framed POVM identical to that of a generalized fusion frame, described in Section 1.1.

Let $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ be a framed POVM where $\dim \mathcal{H}$ is finite. The finite-dimensional assumption on $\mathcal{H}$ allows definition of a real-valued Borel measure $\mu(\omega) = \mathrm{Tr}(M(\omega))$ on the Borel sets of $\Omega$. This positive regular Borel measure is a key element in the following *Radon-Nikodym theorem* for POVMs (see [6]).

**Theorem 3.** *Let $M : \mathcal{B}(\Omega) \to \mathcal{P}(\mathcal{H})$ be a POVM with $\mathcal{H}$ finite-dimensional. Then there exists a regular positive real-valued measure $\mu$ on $\mathcal{B}(\Omega)$ and an operator-valued bounded measurable function $r : \Omega \to \mathcal{P}(\mathcal{H})$ such that*

$$M(\omega) = \int_\omega r(t)\,d\mu(t), \qquad \omega \in \mathcal{B}(\Omega).$$

The measure $\mu$ is called the *base measure* of the POVM and $r$ the *Radon-Nikodym derivative* of the POVM $M$ with respect to $\mu$. This representation is useful in facilitating constructions of POVMs when $\mathcal{H}$ is finite-dimensional.

**Corollary 1.** *If $M$ is a framed POVM with frame bounds $A \leq B$, then*

$$A\mathbb{I}_\mathcal{H} \leq \int_\Omega r(t)\,d\mu(t) \leq B\mathbb{I}_\mathcal{H}$$

It is instructive to observe how this Radon-Nikodym theorem manifests in the motivating examples. In particular, this result shows that, when $\mathcal{H}$ is finite-dimensional, framed POVMs correspond exactly to generalized fusion frames as introduced in Section 1.1.

*Example 5. Let $\mathcal{F} = \{\varphi_k : k \in K\}$ be a frame in $\mathcal{H}$. The associated POVM is given by $M(J) = \sum_{k \in J} \Pi_k$ for subsets $J$ of $\Omega = K$. In this case, the operator-valued function $r$ is given by*

$$r(k) = \Pi_k, \qquad k \in K.$$

*In this special case, there is no need for the finite-dimensional restriction on $\mathcal{H}$. A POVM constructed from a frame in this way automatically possesses a Radon-Nikodym derivative with respect to counting measure on the subsets of $K$.*

*Example 6. In the case of a generalized frame $\Phi : \Omega \to \mathcal{H}$ for a Hilbert space $\mathcal{H}$, the associated POVM is given in (1.9). In this case, the operator-valued function is $r(t) = \Pi_{\Phi(t)}$. As in the previous case, a POVM constructed in this way satisfies a Radon-Nikodym theorem with respect to the given measure $\mu$ on $\Omega$ even when $\mathcal{H}$ is not finite-dimensional.*

*Example 7. For a fusion frame $\{(W_k, w_k) : k \in K\}$, $\Omega = K$ and $\mu$ is counting measure on subsets of $K$. The function $r : \mathcal{B}(K) \to \mathcal{P}(\mathcal{H})$ is given by $r(k) = w_k^2 \Pi_{W_k}$, which coincides with the previous observation that the POVM in this case is defined by*

$$M(\omega) = \sum_{k \in \omega} w_k^2 \Pi_{W_k}, \qquad \omega \subset K.$$

*Although the values of $r$ are not projections, they are non-negative multiples of projections. If the counting measure $\mu$ were replaced by $\nu(k) = w_k^2$, then the expression (1.9) for $M$ would become*

$$M(\omega) = \int_\omega \Pi_{W_k} \, d\nu(k), \qquad \omega \subset K,$$

*and the Radon-Nikodym derivative of $M$ with respect to $\nu$ would have true projections as its values.*

A POVM $M : \Omega \to \mathcal{P}(\mathcal{H})$ is *decomposable* if there is an essentially bounded measurable function $r : \Omega \to \mathcal{P}(\mathcal{H})$ and a measure $\mu$ on $\mathcal{B}(\Omega)$ such that

$$M(\omega) = \int_\omega r(t) \, d\mu(t) \qquad \omega \in \mathcal{B}(\Omega).$$

As observed above, if $\dim \mathcal{H}$ is finite, the POVM is decomposable. Further, every POVM arising from a (generalized) frame is decomposable, even when $\mathcal{H}$ is not finite-dimensional. In effect, decomposable framed POVMs correspond to generalized fusion frames as described in Section 1.1, and thus this concept captures the simultaneous generalization of frames to fusion frames and generalized frames.

## 1.10 Conclusions

In this overview, we have set forth the concept of a framed positive operator-valued measure and shown that classical frames, as well as several generalizations of frames, arise as special cases of framed POVMs. We have described how Naimark's theorem for POVMs leads to notions of analysis and synthesis for POVMs that subsume their frame counterparts. We have further discussed how canonical representations of spectral POVMs lead to canonical descriptions of framed POVMs, and that this leads to a notion of a canonical dual POVM analogous to that of the canonical dual of a frame.

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