System Identification of an Inverted Pendulum on a Cart using Compressed Sensing

Manjish Naik*, Douglas Cochran*,
mnaik@asu.edu, cochran@asu.edu
*School of Electrical and Computer Engineering
Arizona State University
Tempe, Arizona 85287

***Abstract*—This paper deals with system identification of a non-linear, dynamic SIMO system of an inverted pendulum on a cart. Using the approach of compressed sensing the Ordinary Differential Equations (ODEs) describing the dynamics of this system are obtained from a time series data of the inputs and outputs. A library of functions is created and the correct coefficients of these functions are identified by applying the *Basis Pursuit* technique. This work modifies and improves the technique developed by Lai, et al. (2011), enabling it to identify complex coupled systems much more accurately and robustly, requiring fewer measurements.**

***Index Terms*—System Identification; Inverted Pendulum; Compressed Sensing; Sparsity; Basis Pursuit**

I. Introduction

The technique of obtaining a mathematical model of a dynamic system based on the measurements of the observable inputs and outputs of the system constitutes the basic concept of *System Identification* [1]. One such dynamic system which has always been of interest to researchers working in the field of system identification is the inverted pendulum on a cart. The reason for this is that the system exhibits non-linear, unstable and non-minimum phase dynamics and it also is an under-actuated system having more degrees of freedom than control inputs. The full state is not always measurable, and so the identification objectives from such limited dynamics is always a challenge. This system is widely used in robotics, control theory, computer control and space rocket guidance systems.

To identify such systems, researchers often try function approximations using mathematical series like Volterra or Wiener [3]-[5]. Taking inspiration from this idea and by exploiting the favorable properties of the field of compressed sensing, a new technique for efficient system identification is devised.

Compressed sensing has gained significant amount of research interest due to its immense application potential [6]-[11]. The Shannon-Nyquist Sampling Theorem suggests that to accurately reconstruct a signal, it should be sampled at a frequency which is twice its bandwidth. As opposed to this, the novel technique of compressed sensing allows the sampling at a frequency much less than this Nyquist frequency. Here, we exploit the compressibility of signals as most natural signals are compressible when expressed in a suitable basis, e.g.: JPEG images are compressible in Discrete Cosine Transform (DCT) and Wavelet basis. Random linear measurements of such a sparse signal expressed in a suitable basis are taken at sub-nyquist frequencies and later optimization techniques are used to reconstruct the original signal from these sets of incomplete measurements.

This project relies on generating a library of functions of the inputs and outputs of the pendulum system using a power series expansion, and then using compressed sensing theory to obtain the correct coefficients of the these functions. The time series data for inputs and outputs are used as the database for this process of sparse reconstruction.

II. The Inverted Pendulum on a Cart

The Inverted Pendulum on a Cart system is shown in Fig 1. This system consists of a rail on which a cart moves and a pendulum hinged on the top of the cart. The cart and pendulum rod are constrained to move within a vertical plane. The cart acceleration acts as a torque on the free-moving pendulum to swing it up. Since the pendulum is exactly centered above the cart, there are no sidelong forces on the rod and it remains balanced upright. But any small disturbance in the motion of the cart for this balanced pendulum shifts it farther away from the upright position, indicating that the upright is an unstable equilibrium point. Also under no external force the oscillations decay and the rod comes to rest at the 0^o position which is also the stable equilibrium, known as the *pendant* position. A force F is applied to the cart which is the *input*

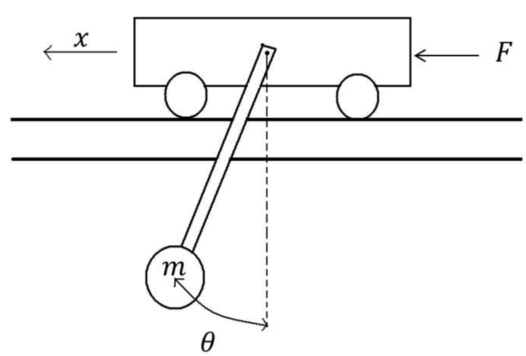

Fig. 1. Physical model of the inverted pendulum on cart system.

to the system. Positive value of F causes the cart to move towards left and negative value to the right. This is due to the

TABLE I
PARAMETERS OF THE PENDULUM SYSTEM

Parameter	Symbol	Value
Mass of Cart	M	0.5 Kg
Mass of pendulum Bob	m	0.275 Kg
Length of Pendulum rod	L	0.5 m
Coefficient of friction between pendulum and pivot	c_p	0.01
Coefficient of friction between cart and track	c_c	0.05

forward and reverse rotations of the motor driving the cart and suitable mechanical amplification of this motion is achieved by a gearing system. The angle θ of the pendulum rod with the vertical is measured and is one of the outputs of the system, while the displacement of the cart y is the other output. Due to the Right Half Plane zeros of this system the typical *inverse response* is seen, where for the cart to move to the right, it must first move to the left and unbalance the pendulum in the correct direction. Also the cart rail has finite length, an additional constraint on the cart motion.

The control objective is to swing up the pendulum to the upright position or the unstable equilibrium and stabilize it there by moving the cart back and forth on the rail.

The parameters of this system and their values are given in Table I. The values of these parameters are selected based on the physical setup in [12].

A. Mathematical Model

After a study of the physical setup of the pendulum on a cart system, detailed mathematical analysis will be performed. [12] gives an excellent step by step derivation of the mathematical modeling of this system. The equation for the cart motion is:

$$(m + M)\ddot{y} = F - c_c\dot{y} - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta \quad (1)$$

The equation for the pendulum motion is:

$$mL^2\ddot{\theta} = -c_p\dot{\theta} - mgL\sin\theta - mL\ddot{y}\cos\theta \quad (2)$$

Equations 1 and 2 describe the model of the cart-and-pendulum motion. It should be noted how these two equations of motion are inter-dependant with each other and this shows the *Coupled* behavior of the system. After substituting the values of the parameters of Table I the following is obtained:

$$\ddot{y} = 1.2903F - 0.0645\dot{y} - 0.1774\ddot{\theta} + 0.1774\dot{\theta}\cos\theta \quad (3)$$

$$\ddot{\theta} = -0.1455\dot{\theta} - 19.6\sin\theta - 2\ddot{y}\cos\theta \quad (4)$$

B. Simulating the System

After obtaining the mathematical equations describing the system, now we implement them in SIMULINK to simulate the behavior of the actual physical system. The purpose of this simulation is to obtain the time-series input-output measurements of the system which will be later used in system identification.

Since this system is highly non-linear and unstable, we use a stabilizing controller initially to stabilize the system in a range and then use a square signal to excite all possible output levels of both outputs of the system. This approach is explained in detail in [15]. The simulation diagram is shown in Figures 2

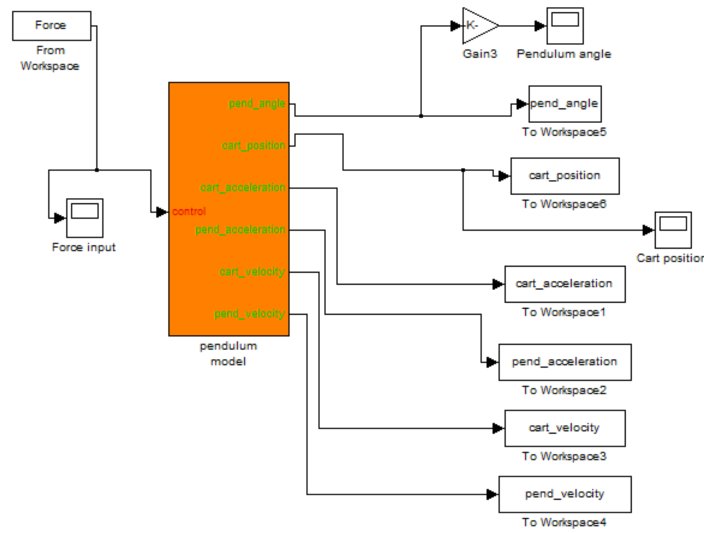

Fig. 2. SIMULINK Block Diagram model of the inverted pendulum on cart system.

and 3. Figure 2 shows the input and output block diagram of the system. The output of the stabilizing controller is fed into the model block and the outputs of pendulum angle and cart position are taken out from it. Also secondary outputs like velocity and acceleration are derived from the primary outputs. Figure 3 shows the internal working of the model

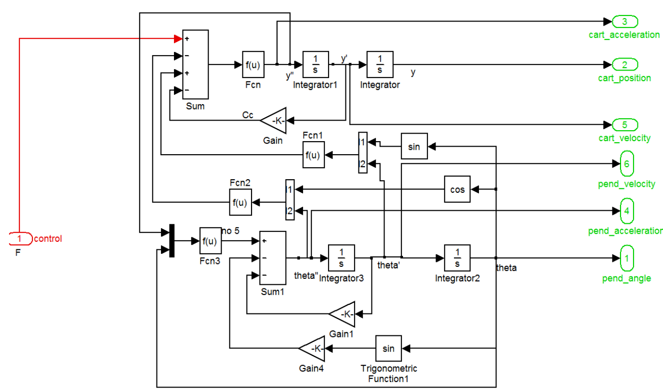

Fig. 3. SIMULINK model of the inverted pendulum on cart system describing the ODEs of the system.

block. It is basically a representation of the ODE's of equations 1 and 2. The input (Force F) is shown in red and primary and secondary outputs are shown in green. Data is generated for 10,000 discrete samples taken at 0.1 KHz sampling frequency and the square input sequence is as shown in Figure 4. The corresponding outputs generated is shown in Figure 5 and 6. Even though these reading are taken over a wide range of

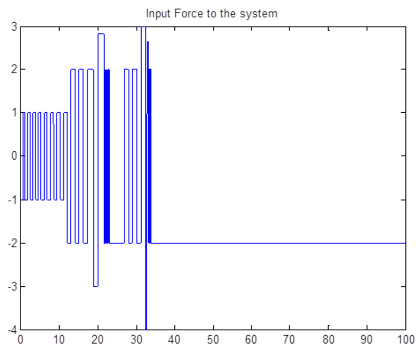

Fig. 4. Force applied to cart sampled at 0.1 KHZ

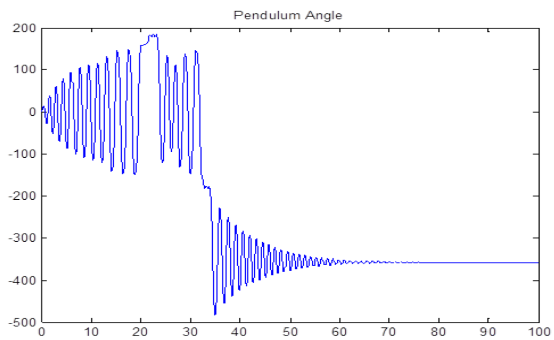

Fig. 5. Angle of pendulum with vertical in degrees.

values for 10,000 samples, we will be selecting only a few out of these for our system identification scheme.

III. System Identification using Compressed Sensing

This section deals with the actual methodology used to identify the inverted pendulum system. We study the original technique proposed by Lai et al. in [13], then we identify some important drawbacks which make it difficult to apply the technique to the problem of System Identification of the Inverted pendulum. After that we draw some inferences and based on these inferences, we propose our modifications to the existing technique. The results of experimentation using the modified technique are presented at the end.

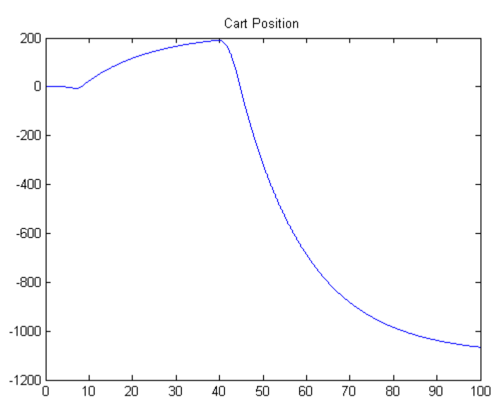

Fig. 6. Position of cart. (negative is towards left and vice versa.)

A. Existing Method by Lai et al.

Consider a non-linear dynamic system which can be represented by the equation $\dot{x} = F(x)$ where x is the state vector containing the states $[x_1, x_2, x_3, \cdots, x_m]^T$ of the system. Lai et al. propose in [13] that the k^{th} component of $F(x)$ can be written as a power series expansion as:

$$[F(x)]_k = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_m=0}^{n} [a_j]_{l_1,l_2..l_m} x_1^{l_1}.x_2^{l_2}.\cdots.x_m^{l_m} \quad (5)$$

where, the coefficient vector $a = [a_j]_{l_1,l_2\cdots l_m}$ is to be determined from the time series data. If the time series data for the states are available at time instants $t_1, t_2 \cdots t_w$ then we can write

$$[F(x(t))]_1 = g(t).a \quad (6)$$

where

$$g(t) = [x_1(t)^0 x_2(t)^0 \cdots x_m(t)^0, x_1(t)^0 x_2(t)^0 \cdots x_m(t)^1, \cdots, x_1(t)^{l_1} x_2(t)^{l_2} \cdots x_m(t)^{l_m})]$$

Now this can be written as a familiar system of linear equations as: $X = G.a$

where $X = [F(x(t))]_1 = [x_1(\dot{t_1}), x_1(\dot{t_2}), x_1(\dot{t_w})]^T$

Therefore:

$$\begin{bmatrix} x_1(\dot{t_1}) \\ x_1(\dot{t_2}) \\ \vdots \\ x_1(\dot{t_w}) \end{bmatrix} = \begin{bmatrix} g(\dot{t_1}) \\ g(\dot{t_2}) \\ \vdots \\ g(\dot{t_w}) \end{bmatrix} (a) \quad (7)$$

Thus we construct the matrix G from the power series expansion where the columns denote the various terms of the power series and the rows denote the time series evaluations of these terms at a particular time instant, t_k.

B. The Modified Technique

Given this background we see two main drawbacks of such a scheme: Firstly, most dynamic systems especially mechanical systems always have a forcing input or excitation term u (e.g. the force which drives the cart in the inverted pendulum system) and the generalized system equation is represented as $\dot{x} = F(x, u)$. Secondly, for coupled systems, we cannot represent $F(x, u)$ as in Equation 5. The reasons for this will become clear after considering the inverted pendulum system and trying to decouple its equations.

After substituting equation 4 in equation 3 and vice versa the following is obtained:

$$\ddot{y} = \frac{1.29F - 0.0645\dot{y} + 0.0257\dot{\theta}\cos\theta + 3.477\sin\theta\cos\theta + 0.1774\dot{\theta}^2\sin\theta}{1 - 0.3548\cos\theta^2} \quad (8)$$

$$\ddot{\theta} = \frac{-0.145\dot{\theta} - 19.6\sin\theta - 2.58F\cos\theta + 0.129\dot{y}\cos\theta - 0.3548\dot{\theta}^2\sin\theta\cos\theta}{1 - 0.3548\cos\theta^2} \quad (9)$$

Converting these two ODE's to a standard representation $\dot{x} = F(x, u)$.
The state vector $x = [x_1, x_2, x_3, x_4]^T = [y, \dot{y}, \theta, \dot{\theta}]^T$ is:

$$\dot{x}_1 = x_2 \quad (10)$$

$$\dot{x}_2 = \frac{1.29u - 0.0645x_2 + 0.0257x_4 \cos x_3 + 3.477 \sin x_3 \cos x_3 - 0.1774x_4^2 \sin x_3}{1 - \cos^2 x_3} \quad (11)$$

$$\dot{x}_3 = x_4 \quad (12)$$

$$\dot{x}_4 = \frac{-0.145x_4 - 19.6 \sin x_3 - 2.58u \cos x_3 + 0.129x_2 \cos x_3 - 0.3548x_4^2 \sin x_3 \cos x_4}{1 - \cos^2 x_3} \quad (13)$$

Thus it is evident that after decoupling the system $\dot{x} = F(x, u)$ cannot be expressed using a power series as $\dot{x}_2$ and $\dot{x}_3$ have the denominator term in it.

Now after establishing these two main drawbacks, its important to realize that the methodology of Lai et al. needs considerable modification in order to be applied to reconstruct the differential equations of the Inverted Pendulum system. We now take a different approach to solve this problem. We first lay the groundwork by some preliminary inferences:
1) Since it's a mechanical system, the ODE will have maximum order of 2. This is a reasonable assumption considering that the main physical quantities involved are displacement, velocity and acceleration.
2) There will be two ODE's to describe the system completely. This also is reasonable considering there are two motions in space: the motion of the cart and the motion of the pendulum.
3) Since it is a system exhibiting oscillatory behavior the power series expansion will have sinusoids of θ along with the polynomial terms.

Considering these inferences we now develop our library of basis functions and the terms involved in the power series expansion. They are as follows: $G = [y, \dot{y}, \ddot{y}, \theta, \dot{\theta}, \ddot{\theta}, \sin\theta, \cos\theta, F]$ where y is the displacement of the cart, θ is the angle between the pendulum and the vertical in degrees and F is the force applied to the cart or the excitation signal. The power series now looks like:

$$\ddot{y} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_9=0}^{n} [a_{y_j}]_{l_1,l_2..l_9} y^{l_1} \dot{y}^{l_2} \ddot{y}^{l_3} \theta^{l_4} \dot{\theta}^{l_5} \ddot{\theta}^{l_6} sin^{l_7}\theta cos^{l_8}\theta F^{l_9} \quad (14)$$

$$\ddot{\theta} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_9=0}^{n} [a_{\theta j}]_{l_1,l_2..l_9} y^{l_1} \dot{y}^{l_2} \ddot{y}^{l_3} \theta^{l_4} \dot{\theta}^{l_5} \ddot{\theta}^{l_6} sin^{l_7}\theta cos^{l_8}\theta F^{l_9} \quad (15)$$

We take the maximum power of the expansion, $n = 2$. Also it is important to note that when we construct our G matrix using these terms, we have to eliminate the column representing the term $\ddot{y}$ from equation 14 and the term $\ddot{\theta}$ from the equation 15 for obvious reasons. Thus we get the linear system of equations $\ddot{y} = G.a_y$ and $\ddot{\theta} = G.a_\theta$

Now using compressed sensing we reconstruct the coefficient vectors a_y and a_θ. It is important to note that these coefficient vectors are sparse due to the fact that only a few terms amongst all the terms of the power series expansion in G will form the ODE for $\ddot{y}$ and $\ddot{\theta}$. In order to get accurate reconstruction from these two systems of linear equations using compressed sensing, G should satisfy the Restricted Isometry Property [11]. We ensure this by normalizing the columns of G by diving each element in the column with the $L2$ norm of the column as explained in [13]. We obtain the sparse reconstruction of these coefficient vectors using the Basis Pursuit technique as:
$min||a_y||_{l1}$ subject to, $G.a_y = \ddot{y}$ and
$min||a_\theta||_{l1}$ subject to, $G.a_\theta = \ddot{\theta}$

IV. Results

The technique was tested for the data set obtained from simulation, by varying two main criteria: number of samples used for reconstruction and the initial condition. Tables II and III show the results. It should be noted that the tables show only the values of non-zero reconstructed coefficients. All other coefficients were found to be zero or extremely small and ignored. It is seen that for different initial conditions, the technique is still able to obtain the accurate reconstructed equation, which shows us that the technique is highly robust. Also only 45% samples are sufficient for accurate reconstruction, which is a very important advantage over other system identification methods using Neural Networks which need large number of measurements for training.

TABLE II
RECONSTRUCTED COEFFICIENTS (ZERO INITIAL CONDITIONS)

LHS	Term	Original Coefficients	Reconstructed Coefficients
$\ddot{y}$	F	1.2903	1.113
	$\dot{y}$	-0.0645	-0.0601
	$\ddot{\theta}$	-0.1774	-0.1701
	$\dot{\theta}\cos\theta$	0.1774	0.1381
$\ddot{\theta}$	$\dot{\theta}$	-0.1455	-0.00973
	$\sin\theta$	-19.6	-19.23
	$\ddot{y}\cos\theta$	-2	-1.905

Note. This experiment was performed for the pendulum to be at pendant position and an initial force of -1 units given to the cart. The number of samples needed for accurate reconstruction were 375 and the number of unknown terms or the size of a_y or a_θ was 768. Therefore 49% samples were needed as compared to the unknowns.

TABLE III
RECONSTRUCTED COEFFICIENTS (NON-ZERO INITIAL CONDITIONS)

LHS	Term	Original Coefficients	Reconstructed Coefficients
$\ddot{y}$	F	1.2903	1.2481
	$\dot{y}$	-0.0645	-0.0272
	$\ddot{\theta}$	-0.1774	-0.1739
	$\dot{\theta}\cos\theta$	0.1774	0.1691
$\ddot{\theta}$	$\dot{\theta}$	-0.1455	-0.00130
	$\sin\theta$	-19.6	-19.4021
	$\ddot{y}\cos\theta$	-2	-1.89

Note. This experiment was performed for the pendulum to be at 28^o position and an initial force of -1 units given to the cart. The number of samples needed for accurate reconstruction were 350 and the number of terms to reconstruct or the size of a_y or a_θ was 768. Therefore 45% samples were needed as compared to the unknowns.

V. Conclusion

System Identification is an extremely important and widely used field and the inverted pendulum systems have always been used as benchmarks to analyze the efficacy of identification techniques. This paper gives a new and modified technique for system identification and applies it to a SIMO, non-linear, coupled system. We rectify two main drawbacks in the existing technique, which enables us to apply it to a highly complex problem. In order to do this, first we simulate the pendulum and cart system and construct the excitation signal based on a stabilizing controller. Then we obtain the time-series input-output measurement data for this system. The power series expansion is then formulated to include a library of functions and later we use the compressed sensing approach to reconstruct the terms of the power series expansion from the input-output data. This technique is highly robust and can reconstruct the ODE's using very few samples as compared to the unknowns. Also it is able to reconstruct two coupled ODE's with high accuracy.

This modified technique will prove to be extremely useful and its success for the inverted pendulum problem is a promising start.

VI. Future work

This paper hopes to stimulate future research in certain similar areas. There are some issues which are being currently worked on. A few will be provided in this section.

Currently we are focusing on sinusoids of single frequency for our power series expansion, i.e. $\sin\theta$. But for its application to a broader class of systems, we should find a way to incorporate other frequencies as well and include $\sin\omega\theta$ in our basis functions. This will increase the number of terms to be reconstructed and will in turn increase computational complexity.

Also, the columns of the G matrix may be highly correlated in some scenarios where two functions approximate each other in a given range of values. Example: θ and $\sin\theta$ are almost equal for small values of θ. In this case it becomes extremely hard for the optimizer to differentiate between the correlated columns and reconstruct the coefficients. A method to decorrelate the columns is essential.

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