On Allocating Interconnecting Links against Cascading Failures in Cyber-Physical Networks

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***Abstract*—We consider a cyber-physical system consisting of two interacting networks, e.g., a cyber-network overlaying a physical-network. It is envisioned that these systems are more vulnerable to attacks since node failures in one network may result in (due to the interdependence) failures in the other network causing a cascade of failures that would potentially lead to the collapse of the entire infrastructure. Robustness of interdependent systems against this sort of catastrophic failure hinges heavily on the allocation of the (interconnecting) links that connect nodes in one network to nodes in the other network. In this paper, we characterize the *optimum* inter-link allocation strategy in the case where no information regarding the intra-structures of the networks is available. Namely, we develop a "regular" allocation strategy that allots inter-network links uniformly across all nodes and show both analytically and experimentally that this regular allocation of inter-network edges yields a better performance (from a network resilience perspective) compared to the other possible strategies; e.g., deterministic non-uniform strategy, random allocation strategy, etc.**

I. INTRODUCTION

Today's worldwide network infrastructure is built upon a large family of interacting cyber-networks (e.g., the Internet) and physical systems (e.g., the power grid). There is a consensus that integrated cyber-physical systems will emerge as the underpinning technology for major industries in the 21st century [1]. The smart grid is one archetypal example of such systems. In the event of attack on one of these networks, the interdependence can lead to multiple nodes (in both the physical system and the cyber-network) being compromised and this may result in a cascade of failures. The consequent impact on the overall cyber-physical system can be catastrophic because the affected area can be much greater than what would be compromised by the same attack in the absence of network interdependence. For instance, the cascading effect of even a partial Internet blackout could disrupt major national infrastructure networks involving Internet services, power grids and financial markets [2].

Despite recent studies of cascading failures in complex networks, the dynamics of cascading failures and the impact of such failures across multiple networks are not well understood. There is thus a need to develop a new network science for modeling and quantifying cascading failures, and to develop network management algorithms that improve network robustness and ensure overall network reliability against cascading failures. Most existing studies on failures in complex networks consider single networks only. A notable exception is the very recent work [2] in which a "one-to-one correspondence" model for studying the ramifications of interdependence between two networks is set forth. This model assumes that two random networks, say network A and network B, have the same number of nodes and that each node in A has a bi-directional *inter-edge* connecting it to a *unique* node in B. Further, it is assumed that a node in either network can function *only if* it has support from the other network; i.e., it is connected (via an inter-edge) to at least one functioning node of the other network. As in the case of a single network [3], [4], the random removal of a fraction $(1-p)$ of the nodes in A would lead to a cascade of failures. It turns out that there exists a critical threshold on p, denoted by p_c, above which the entire system still contains a mutually connected functioning giant component [2] despite the cascading effect; on the other hand if $p < p_c$, both networks go into a complete fragmentation and the entire system collapses. It is further observed in [2] that this kind of interdependent network systems have a much larger p_c compared to that required in the individual constituent networks. To mitigate this vulnerability issue, [5] proposed a "random" allocation strategy in which each node in a network is supported by a random number of inter-edges, which are i.i.d. across all nodes. More inter-edges should, of course, lead to higher robustness against cascading failures. What is unclear is how effectively the random allocation mechanism utilizes the limited resources, since this strategy can yield a non-negligible fraction of nodes in both networks having no support from the other network.

In this study, we focus on the situation where no information about intra-degree distributions is available. We propose a "regular" allocation strategy for allotting inter-edges, where each node is uniformly supported by exactly k inter-edges. In contrast to random allocation, this regular allocation scheme guarantees each node in both networks a prescribed level of support in terms of inter-edges. As anticipated, we find that this regular allocation scheme indeed yields dramatic improvement of resilience to cascading failures. Moreover, we show both analytically and experimentally that the regular allocation can always lead to higher robustness than any unequal allocation strategy. We believe that our findings along this line shed light on the design of interdependent systems.

II. SYSTEM MODEL

We consider a cyber-physical system consisting of two interacting networks, say network A and network B. Both networks are assumed to have N nodes and the vertex

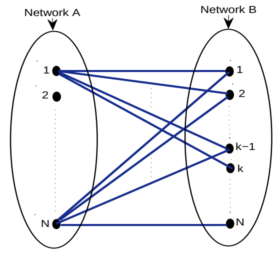


Fig. 1. *The* regular *allocation strategy of inter-edges is depicted. Each node in A is connected to exactly k nodes in B, and vice versa.*

sets in their respective graphical representations are denoted by $\{v_1, \ldots, v_N\}$ and $\{v'_1, \ldots, v'_N\}$. We refer to the edges connecting nodes within the same network as *intra-edges* and those connecting nodes from two different networks as *inter-edges*. Simply put, we assume that a node can function *only if* it is connected (via an inter-edge) to at least one functioning node in the other network [2]; see below for a more precise definition. Clearly, the interdependency between two networks is intimately related to the inter-edges connecting them. Therefore, we find it convenient to introduce an $N \times N$ interdependency matrix $\mathbf{C}$ that represent the bi-directional inter-edges between A and B. Namely, for each $n, m = 1, \ldots, N$, we let

$$(\mathbf{C})_{nm} = \begin{cases} 1 & \text{if } v_n \text{ and } v'_m \text{ are interdependent} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

As mentioned before, here we consider the *regular* allocation strategy of the inter-edges where each node in network A and network B has exactly k inter-edges; k is taken to be an integer satisfying $k \leq N$. Without loss of generality, this strategy can be implemented in the following manner: For each $n = 1, 2, \ldots, N$, let the interdependency matrix be given by

$$(\mathbf{C})_{nm} = \begin{cases} 1 & \text{if} \quad m = n, n \oplus 1, \ldots, n \oplus (k-1) \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where we define $n \oplus l = n + l - \mathbf{1}[n + l > N] \cdot N$; see also Figure 1.

We are interested in evaluating the network robustness in the case of random node failures (or equivalently random attacks). Specifically, in the dynamics of cascading failures, we assume that a node is *functioning* at Stage i if the following conditions are satisfied [2], [5]: i) The node has at least one inter-edge with a node that was functioning at Stage $i-1$; ii) The node belongs to the giant (i.e., the largest) component of the sub-network formed by the nodes (of its own network) that satisfy condition i). Also, for both networks, a giant component that is composed of functioning nodes will be referred to as a *functioning giant component*.

We assume that the cascade of failures is triggered by the failure of a fraction $1-p$ of the nodes in network A. We further assume that these $(1-p)N$ nodes are chosen (say by the attacker) uniformly at random amongst all nodes in network A. By the definitions given above, we can see that after this initial attack, only nodes in the functioning giant component of A can operate properly. As a result of that, in the next stage, some of the nodes in network B may end up losing all of their inter-connections and turn dysfunctional. In that case, the nodes that can function properly in network B will only be those who are in the functioning giant component of B. But, this fragmentation of network B may now trigger further failures in network A due to nodes that lose all their B-connections. In this manner, the cascade of failures propagates alternatively between A and B, eventually leading to either 1) *residual functioning giant components in both networks*, or 2) *complete failure of the entire system*.

We note that the proposed regular allocation strategy uniformly allots the inter-edges across the nodes, and does not require a priori information of intra-structures of the networks. It is expected that the impact of this strategy on network robustness would depend on specific intra-degree distributions. Nevertheless, we show that irrespective of the actual intra-structures of the networks, the regular allocation strategy is superior (in terms of network robustness) to any other strategy if no information regarding these structures is a priori available.

TABLE I
KEY NOTATION IN THE ANALYSIS OF CASCADING FAILURES

A_i, B_i	the functioning giant components in A and B at stage i
p_{Ai}, p_{Bi}	the fractions corresponding to functioning giant components at stage i, $\lvert A_i\rvert = p_{Ai}N$, $\lvert B_i\rvert = p_{Bi}N$
$\bar{A}_i$, $\bar{B}_i$	the remaining nodes in A and B retaining at least one inter-edge at stage i.

III. Analysis of Cascading Failures

In this section, we analyze the dynamics of cascading failures in two interacting networks. A principal objective of this study is to quantify the effectiveness of regular allocation strategy on network robustness via i) characterizing the size of the remaining giant components in A and B after the cascade has reached to a steady state, and ii) finding the corresponding critical threshold p_c. To that end, we will use the technique of generating functions [6], [7] to analyze the sizes of functioning giant components in two networks at each stage. For convenience, the notation that will be used in the calculations are summarized in Table I.

A. Stage 1 : Random failure of nodes in network A

Following the failures of a fraction $1-p$ of randomly selected nodes in A, the remaining network $\bar{A}_1$ has size pN; since we eventually let N grow large, pN can be approximated as an integer. As in [2], [5], [6], [7], we use the technique of generating functions to quantify the fraction of the functioning giant component $A_1 \subset \bar{A}_1$. Specifically, let the function $P_A(p)$ determine the fraction of the giant component in a random subgraph that occupies p fraction of the nodes in network A (The exact calculation of $P_A(p)$ will be elaborated later). It follows that the functioning giant component has size

$$|A_1| = pP_A(p)N = p_{A1}N. \quad (3)$$

As shall become apparent soon, at the end of each stage it is necessary to determine not only the size of the functioning giant component, but also the specific inter-edge distribution over the functioning nodes, i.e., the number of functioning nodes having different number of inter-edges. Here, at the end of Stage 1, each node in A_1 has still k inter-edges from network B since B has not changed yet.

B. Stage 2 : Effect of random A-node failures on network B

As the functional part of network A fragments from A to A_1 (in Stage 1), some of the inter-edges that were supporting B-nodes will be removed. Observe that the probability of removal can be approximated by $1 - |A_1|/|A| = 1 - p_{A1}$ for each inter-edge. With this perspective, a B-node loses $k - j$ of its inter-edges with probability $\binom{k}{j} p_{A1}^j (1 - p_{A1})^{k-j}$. Moreover, it stops functioning with probability $(1 - p_{A1})^k$ due to losing all of its k inter-edges. As a result, with $\bar{B}_2$ denoting the set of nodes in B that retain at least one inter-edge, we have

$$|\bar{B}_2| = \left(1 - (1 - p_{A1})^k\right) \cdot N = p'_{B2} N, \tag{4}$$

where $p'_{B2} = 1 - (1 - p_{A1})^k$. Also, the distribution of inter-edges over the nodes in $\bar{B}_2$ is given by

$$|\bar{B}_2|_j = \binom{k}{j} p_{A1}^j (1 - p_{A1})^{k-j} \cdot N, \quad j = 1, 2 \ldots, k, \tag{5}$$

with $|\bar{B}_2|_j$ denoting the number of nodes in $\bar{B}_2$ that have j inter-edges.

As in Stage 1, the size of the functioning giant component $B_2 \subset \bar{B}_2$ can be predicted by:

$$|B_2| = p'_{B2} P_B(p'_{B2}) N = p_{B2} N, \tag{6}$$

where $P_B(\cdot)$ is defined analogously to the definition of $P_A(\cdot)$ given in Section III-A. Obviously, each node in $\bar{B}_2$ can survive as a functioning node in B_2 with probability $P_B(p'_{B2})$. Thus, for each $j = 1, 2, \ldots, k$, the number of nodes in B_2 that has j inter-edges is given (in view of (5)) by

$$|B_2|_j = P_B(p'_{B2}) \binom{k}{j} p_{A1}^j (1 - p_{A1})^{k-j} \cdot N. \tag{7}$$

C. Stage 3 : Further fragmentation of A due to B-node failures

Due to the fragmentation of the functional part of network B from $\bar{B}_2$ to B_2 (not B to B_2), some of the nodes in A_1 may now lose all their inter-edges and stop functioning. To compute the probability of this event, first observe that each inter-edge from $\bar{B}_2$ to A_1 will be removed with an approximate probability of $1 - |B_2|/|\bar{B}_2| = 1 - P_B(p'_{B2})$. Hence the probability that a node in A_1 will lose all of its inter-edges is given by $(1 - P_B(p'_{B2}))^k$. It also follows that the size of the network $\bar{A}_3 \subset A_1$ that constitutes the nodes that did not lose all their inter-connections is given via

$$|\bar{A}_3| = p_{A1} \left(1 - (1 - P_B(p'_{B2}))^k\right) \cdot N. \tag{8}$$

In other words, from A_1 to $\bar{A}_3$, a fraction $1 - |\bar{A}_3|/|A_1| = (1 - P_B(p'_{B2}))^k$ of the nodes have failed. As usual, the next step is now to compute the functioning giant component $A_3 \subset \bar{A}_3$. However, this a challenging task as noted in [2]. Instead, we view the joint effect of the node failures in Stage 1 and Stage 3 as equivalent (in terms of the size of the resulting functional giant component, i.e., $|A_3|$) to the effect of an initial random attack which targets an appropriate fraction of the nodes that will later be determined. Intuitively speaking, the node failures in A_1 at Stage 3, i.e., the removal of a fraction $(1 - P_B(p'_{B2}))^k$ of nodes from A_1, has the same effect as taking out the same portion from $\bar{A}_1$ [2]. In other words, it is equivalent to the removal of a fraction $p(1 - P_B(p'_{B2}))^k$ of the nodes from A. Recalling also that a fraction $1 - p$ of the nodes were already failed (as a result of the initial attack) at Stage 1, we find that the fragmentation of A to $\bar{A}_3$ can as well be modeled (only with respect to the size of A_3) by an initial attack targeting a fraction

$$1 - p + p\,(1 - P_B(p'_{B2}))^k = 1 - p\left(1 - (1 - P_B(p'_{B2}))^k\right)$$

of the nodes. It is now a standard step to conclude that with $p'_{A3} = p\left(1 - (1 - P_B(p'_{B2}))^k\right)$ the size of the functioning giant component A_3 is given by

$$|A_3| = p'_{A3} P_A(p'_{A3}) N = p_{A3} N. \tag{9}$$

D. Stage 4 : Further fragmentation of B

Due to the network fragmentation from $\bar{A}_3$ to A_3 in Stage 3, each inter-edge supporting a B_2-node will be disconnected with the probability that equals the proportion of the failed fraction from $\bar{A}_3$ to A_3, i.e., $1 - |A_3|/|\bar{A}_3| = 1 - P_A(p'_{A3})/P_A(p)$ by (8) and (9). Consequently, a node in B_2 with j inter-edges will stop functioning with probability $(1 - P_A(p'_{A3})/P_A(p))^j$. Recalling also the inter-edge distribution (7), the fraction L of node failures in B_2 is given by

$$\begin{aligned}
L &= \sum_{j=1}^{k} |B_2|_j \left(1 - \frac{P_A(p'_{A3})}{P_A(p)}\right)^j \\
&= P_B(p'_{B2}) \sum_{j=1}^{k} \binom{k}{j} p_{A1}^j (1 - p_{A1})^{k-j} \left(1 - \frac{P_A(p'_{A3})}{P_A(p)}\right)^j \\
&= P_B(p'_{B2}) \left(\left(1 - p_{A1} \frac{P_A(p'_{A3})}{P_A(p)}\right)^k - (1 - p_{A1})^k\right) \\
&= P_B(p'_{B2}) \left((1 - pP_A(p'_{A3}))^k - (1 - p_{A1})^k\right).
\end{aligned}$$

Since $|\bar{B}_4| = |B_2| - L \cdot N$, we now obtain

$$|\bar{B}_4| = P_B(p'_{B2}) \left(1 - (1 - pP_A(p'_{A3}))^k\right) \cdot N. \tag{10}$$

In order to compute the size of the functioning giant component $B_4 \subset \bar{B}_4$ we proceed as in Stage 3: We view the joint effect of node removals in Stage 2 and Stage 4 as equivalent to that of an initial random attack which targets an appropriate fraction of the nodes. To determine this fraction, first observe that the failures in Stage 3 have triggered further node failures in B_2 resulting a fraction

$$1 - |\bar{B}_4|/|B_2| = 1 - \left(1 - (1 - pP_A(p'_{A3}))^k\right)/p'_{B2} \tag{11}$$

of the nodes' failure. Next, note that the effect of these failures on $|B_4|$ is equivalent to that of taking out the same fraction of nodes from $\bar{B}_2$ [2]. Moreover, it has the same effect as taking out a fraction $p'_{B2}\left\{1 - \left(1 - (1 - pP_A(p'_{A3}))^k\right)/p'_{B2}\right\}$ of the nodes in B. Now, recalling that a fraction $1 - p'_{B2}$ of nodes in

B have failed in Stage 2, we conclude that the joint effect of cascading failures in Stage 2 and Stage 4 (on $|B_4|$) is identical to that of an initial random attack which targets a fraction

$$1-p'_{B2}+p'_{B2}\left(1-\frac{1-(1-pP_A(p'_{A3}))^k}{p'_{B2}}\right)=(1-pP_A(p'_{A3}))^k$$

of nodes. As usual, with $p'_{B4}=1-(1-pP_A(p'_{A3}))^k$ we conclude that the size of the functioning giant component B_4 is given by $|B_4|=p'_{B4}P_B(p'_{B4})N=p_{B4}N$.

E. Cascading dynamics of node failures

As mentioned earlier, the main goal of this section is to characterize the size of the functional giant components at the steady state. Indeed, along the lines outlined above, one can obtain the size of all functioning giant components $A_1\supset A_3\supset\ldots\supset A_{2m+1}$ and $B_2\supset B_4\supset\ldots\supset B_{2m}$ for any integer m. However, it is easy to observe the pattern in the expressions obtained so far and conclude that with $p'_{A1}=p$ the size of all giant components are given by the recursive relations:

$$\begin{cases} p_{Ai}=p'_{Ai}P_A(p'_{Ai}), \\ p'_{Ai}=p\left(1-(1-P_B(p'_{Bi-1}))^k\right), \end{cases} \quad i=1,3,5,\ldots \tag{12}$$

$$\begin{cases} p_{Bi}=p'_{Bi}P_B(p'_{Bi}), \\ p'_{Bi}=1-\left(1-pP_A(p'_{Ai-1})\right)^k, \end{cases} \quad i=2,4,6,\ldots. \tag{13}$$

This recursive process stops at an "equilibrium point" where we have $p'_{B2m-2}=p'_{B2m}$ and $p'_{A2m-1}=p'_{A2m+1}$ so that neither network A nor network B fragments further. Setting $x=p'_{A2m+1}$ and $y=p'_{B2m}$, this yields the set of equations

$$x=p\left(1-(1-P_B(y))^k\right) \quad y=1-(1-pP_A(x))^k. \tag{14}$$

Furthermore, the fraction of nodes in the giant components are given by $P_{A_\infty}=xP_A(x)$ and $P_{B_\infty}=yP_B(y)$.

The analysis carried out up to this point is valid for all networks irrespective of their intra-structures. To get a better sense of the results, we now study a simple case where network A and network B are Erdős-Rényi networks with average intra-degrees a and b, respectively. In this case, with $0\le x,y\le 1$, we have [7] $P_A(x)=1-f_A$ and $P_B(y)=1-f_B$, where f_A, f_B are the unique solutions of

$$f_A=\exp\{ax(f_A-1)\} \quad \text{and} \quad f_B=\exp\{by(f_B-1)\}. \tag{15}$$

Reporting these into (14), we get

$$x=p(1-f_B^k) \qquad y=1-(1-p(1-f_A))^k, \tag{16}$$

whence the giant component fractions are given by

$$\begin{aligned} P_{A_\infty}&=p(1-f_B^k)(1-f_A) \\ P_{B_\infty}&=\left(1-(1-p(1-f_A))^k\right)(1-f_B). \end{aligned} \tag{17}$$

Next, reporting (16) into (15) we find $f_A=\exp\{ap(1-f_B^k)(f_A-1)\}$ and $f_B=\exp\{b\left(1-(1-p(1-f_A))^k\right)(f_B-1)\}$, which is also equivalent to

$$\begin{aligned} f_B&=\sqrt[k]{1-\frac{\log f_A}{(f_A-1)ap}} \quad \text{if } 0\le f_A<1; \quad \forall f_B \text{ if } f_A=1 \\ f_A&=1-\frac{1-\sqrt[k]{1-\frac{\log f_B}{(f_B-1)b}}}{p} \quad \text{if } 0\le f_B<1; \quad \forall f_A \text{ if } f_B=1 \end{aligned} \tag{18}$$

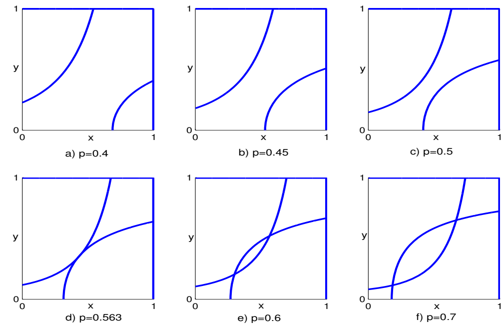


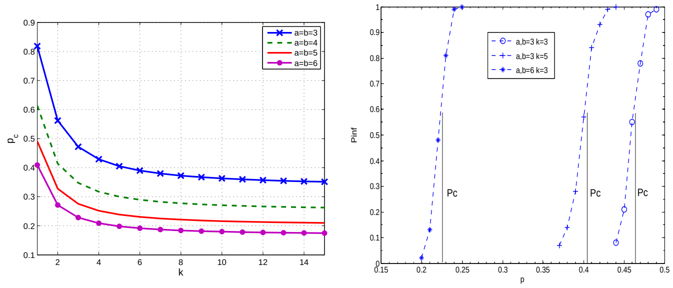
Fig. 2. *$a=b=3$, $k=2$. x-axis represents f_A and y-axis represents f_B.*

We note that the system of equations (18) has always a trivial solution $f_A=f_B=1$, in which case the (steady state) functional giant component has zero fraction for both networks. More interesting cases arise for large values of p when there exists non-trivial solutions to (18). In particular, we focus on determining the critical threshold p_c, i.e., the minimum p that yields a non-trivial solution of the system. In general, it is not possible to derive an explicit expression for p_c. Instead, we can solve (18) graphically for a given set of parameters a,b,k,p and infer the critical threshold p_c from the plots. For instance, Figure 2 shows the possible solutions of the system for several different p values when $a=b=3,k=2$. In Figure 2 (a-c), when $p<p_c$, there is only the trivial solution $f_A=f_B=1$, i.e., both networks go into a complete fragmentation. In Figure 2(d), when $p=p_c$, there exist one non-trivial solution, since the two curves intersect tangentially at one point. In Figures 2(e-f), when $p>p_c$, there exist two non-trivial points corresponding to two sets of giant component sizes. In these cases, the solution corresponding to the cascading failures should be the point closer to the origin. In other words, the cascading failures stops when it reaches the larger giant component (see (17)).

In general, the system of equations through which f_{A_c}, f_{B_c} and p_c can be computed for any given a,b,k is as given below:

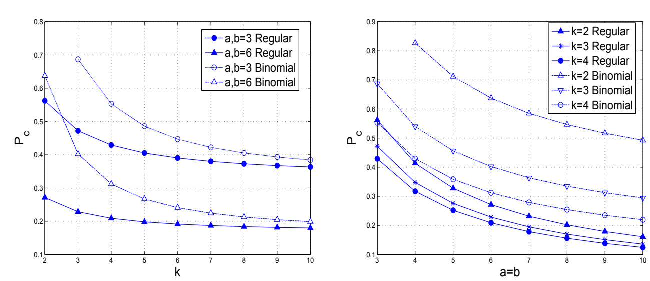
$$\begin{aligned} f_{B_c}&=\sqrt[k]{1-\frac{\log f_A}{(f_A-1)ap}} \quad \text{if } 0\le f_A<1; \ \forall f_B \text{ if } f_A=1 \\ f_{A_c}&=1-\frac{1-\sqrt[k]{1-\frac{\log f_B}{(f_B-1)b}}}{p} \quad \text{if } 0\le f_B<1; \ \forall f_A \text{ if } f_B=1 \\ &\frac{df_A}{df_B}\Big|_{f_A=f_{A_c},f_B=f_{B_c}}\times\frac{df_B}{df_A}\Big|_{f_A=f_{A_c},f_B=f_{B_c}}=1 \end{aligned} \tag{19}$$

The analysis results are now corroborated by simulations. In Figure 3(a), we show the variation of p_c with respect to k for different values of $a=b$, where the critical p_c values are obtained by solving the system (19) graphically. To verify these findings, we pick a few sets of values a, b and k from the curves in Figure 3(a) and run simulations with $N=5000$ nodes to estimate the probability p_{inf} of the existence of a functional giant component in steady state. As expected [2], in all curves we see a sharp increase in p_{inf} as p approaches to a critical threshold p_c. It is also clear from Figure 3(b) that for all parameter sets such sharp transition occurs when p is close to the corresponding p_c value given in Figure 4(a).



(a) (b)

Fig. 3. *a) The critical p_c values obtained by solving the system (19) graphically. b) Simulation results for the regular allocation strategy when $N = 5000$. The p_{inf} has a sharp increase when p approaches to a critical threshold p_c; for $(a = b = 3, k = 3)$, $(a = b = 3, k = 5)$ and $(a = b = 6, k = 3)$, the critical p_c values are roughly equal to 0.465, 0.405 and 0.225, respectively. Clearly, these p_c values are in perfect agreement with the corresponding ones of Figure 4(a) that were obtained analytically.*



(a) (b)

Fig. 4. *The critical values of p_c with average inter-degree equal to k. a) p_c v.s. k for different values of $a = b$; b) p_c v.s. $a = b$ for various k values. In all cases, the regular inter-degree distribution yields a smaller p_c than the binomial distribution with the same mean value k.*

IV. Random Allocation vs Regular Allocation

As a firs step in showing the optimality of the regular allocation strategy, we now compare its robustness against the random allocation strategy introduced in [5]. According to the random allocation strategy, each node has a *random* number of inter-edges which is distributed binomially. To compare these strategies in a fair manner, consider two Erdős-Rényi networks of the same size N, with intra-degrees a and b. As before, let the number of inter-edges per node be fixed at k in the regular allocation strategy, whereas let the inter degree of any node in the random allocation strategy be a binomial random variable with nk trials and success probability $\frac{1}{k}$ [5]; clearly this ensures that the expected inter-degree of a node in both strategies are equal. The critical threshold values p_c corresponding to both strategies are compared under a variety of conditions. Figure 4(a) depicts p_c as a function of mean inter-degree k, for various values of $a = b$, whereas Figure 4(b) depicts the variation of p_c with respect to $a = b$ for different k values. It can be seen that the regular allocation strategy yields a much smaller p_c (and thus, a more robust system) than the random allocation strategy in all cases. For example, for $a = b = k = 4$, the Binomial allocation [5, Figure2] gives $p_c = 0.43$, whereas as seen via Figure 4(a), the regular allocation yields a critical threshold at 0.317. This is a significant difference in that the random allocation strategy ensures a mutually connected functioning giant component despite a random failure of at most 57% of the nodes, whereas the strategy proposed here is resistent to a random failure of up to 68% of the nodes. Indeed, in some cases, our strategy can outperform that in [5] even with half (average) inter-degrees per node. For instance, when $a = b = 4$, our strategy yields $p_c = 0.414$ with only $k = 2$ as compared to $p_c = 0.43$ of the random allocation strategy with $k = 4$.

We believe that the drastic improvement in robustness against cascading failures can be attributed to the following two reasons. First, in the random allocation, there always exists a non-negligible fraction of nodes with no inter-edge support from the other network. Clearly, the regular allocation scheme promises a guaranteed support in terms of inter-edges, for all nodes in both networks. Furthermore, as will be elaborated in Section V, in the absence of the intra-structure information, the regular allocation strategy always yields smaller p_c than any unequal allocation strategy, which corresponds to a specific realization of the random allocation.

V. Optimality of Regular Allocation Strategy

In this section, we show analytically that the regular allocation strategy always yields a better robustness than any unequal allocation strategy and thus it is optimal in the absence of intra-structure information. For convenience, we denote a system that uses the regular allocation strategy as System 1. We start by comparing this system to the following auxiliary System 2: Consider two subgraphs of size αN, namely S_{A_α} and S_{B_α}, whose nodes are randomly selected from the original network A and network B; let also the remaining fractions be denoted as $S^c_{A_\alpha}$ and $S^c_{B_\alpha}$, respectively. We assume that the inter-edges between S_{A_α} and S_{B_α} are built in the same way as System 1, with k' inter-edges per node, while $S^c_{A_\alpha}$ and $S^c_{B_\alpha}$ are connected with one-to-one correspondence; this amounts to allocating inter-edges according to a deterministic non-uniform strategy.

We analyze the dynamics of cascading failures in System 2 by using an iterative approach similar to that in Section III. This time, we find that the fractions of giant components at each stage are given by

$$\begin{aligned} p_{Ai} &= p'_{Ai} P_A(p'_{Ai}), \\ p'_{Ai} &= p\left(\left(1-(1-P_B(p'_{Bi-1}))^{k'}\right)\alpha + P_B(p'_{Bi-1})(1-\alpha)\right), \end{aligned} \tag{20}$$

for $i = 3, 5, 7, \ldots$, and

$$\begin{aligned} p_{Bi} &= p'_{Bi} P_B(p'_{Bi}), \\ p'_{Bi} &= \left(1-(1-pP_A(p'_{Ai-1}))^{k'}\right)\alpha + pP_A(p'_{Ai-1})(1-\alpha), \end{aligned} \tag{21}$$

for $i = 2, 4, 6, \ldots$. It is again important to remind that the above analysis is valid irrespective of the particular intra-structures of the networks.

We now compare the two systems in terms of their robustness against random node failures. For the sake of a fair comparison, network A (resp. network B) of the two systems are assumed to be *statically identical*. For given k and k', let $p_{c_1}(k)$ and $p_{c_2}(k', \alpha)$ define the critical thresholds of System

1 and System 2, respectively. Moreover, for any given p, let $P_{A^1_\infty}(p;k), P_{B^1_\infty}(p;k)$ and $P_{A^2_\infty}(p;k',\alpha), P_{B^2_\infty}(p;k',\alpha)$ define the fractions of functional giant components (of network A and network B, respectively) at the steady-state. The following proposition shows that if the two systems are matched through their mean inter-degrees, i.e., if $k = \alpha k' + 1 - \alpha$, System 1 is always more robust than System 2 against random failures.

Proposition 5.1: For any α in $[0,1]$, we have

$$\begin{aligned} &P_{A^1_\infty}(p;k) \geq P_{A^2_\infty}(p;k',\alpha) \\ &P_{B^1_\infty}(p;k) \geq P_{B^2_\infty}(p;k',\alpha) \\ &p_{c_1}(k) \leq p_{c_2}(k',\alpha). \end{aligned} \tag{22}$$

whenever $k = \alpha k' + 1 - \alpha$.

Proof: Since $P_A(x)$ and $P_B(x)$ are monotonically increasing [7], a sufficient condition ensuring the first two inequalities in (22) to hold is

$$\begin{aligned} p'_{Ai}(p;k) \geq p'_{Ai}(p;k',\alpha), \quad i = 1,3,5\ldots, \\ p'_{Bi}(p;k) \geq p'_{Bi}(p;k',\alpha), \quad i = 2,4,6\ldots, \end{aligned} \tag{23}$$

where $p'_{Ai}(p;k)$, $p'_{Bi}(p;k)$, and $p'_{Ai}(p;k',\alpha)$, $p'_{Bi}(p;k',\alpha)$ denote the fractions in the recursive relations (12)-(13), and (20)-(21), respectively. In order to show (23), set $u_i = 1 - pP_A(p'_{Ai})$, for $i = 1,3,5,\ldots$, and $v_i = 1 - P_B(p'_{Bi})$, for $i = 2,4,6,\ldots$, and observe that (23) will follow upon showing

$$\begin{aligned} u_i^{\alpha k'+1-\alpha} \leq \alpha u_i^{k'} + (1-\alpha)u_i, \quad i = 1,3,5,\ldots \\ v_i^{\alpha k'+1-\alpha} \leq \alpha v_i^{k'} + (1-\alpha)v_i, \quad i = 2,4,6,\ldots. \end{aligned}$$

For any $\alpha \in [0,1]$, both inequalities hold thanks to the convexity of $u^{k'}$ with $u \geq 0$, by appealing to Jensen's inequality. It follows that (23) and the first two inequalities in (22) hold.

We next prove the inequality $p_{c_1}(k) \leq p_{c_2}(k',\alpha)$ by contradiction. Assume that $p_{c_2}(k',\alpha) < p_{c_1}(k)$ and fix p such that $p_{c_2}(k',\alpha) < p < p_{c_1}(k)$. Then, let a fraction $1-p$ of the nodes randomly fail in network A of both systems. Since p is less than the p_{c_1}, the node failures will eventually lead to complete disfunctioning of the two networks in System 1, i.e., we get $P_{A^1_\infty}(p;k) = P_{B^1_\infty}(p;k) = 0$. On the other hand, the fact that p is larger than the critical threshold p_{c_2} ensures $P_{A^2_\infty}(p;k',\alpha) > 0$ and $P_{B^2_\infty}(p;k',\alpha) > 0$ by definition. This clearly contradicts the first two inequalities in (22) and therefore it must hold that $p_{c_1}(k) \leq p_{c_2}(k',\alpha)$. ∎

In the manner outlined above, we can indeed show that the proposed regular allocation strategy always yields a better robustness than any unequal allocation strategy. Specifically, consider a system where the networks are split into ℓ parts and for each $i = 1,\ldots,\ell$, a fraction α_i of the nodes receive k_i inter-edges per node. It is also assumed that $\sum_{i=1}^{\ell} \alpha_i = 1$. This inter-edge allocation strategy corresponds to the general version of the deterministic non-uniform strategy and it can be implemented by first partitioning the networks randomly into ℓ parts with sizes $\alpha_1 N, \ldots \alpha_\ell N$ and then assigning the inter-edges as in the case of System 2. With $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\boldsymbol{k} = (k_1, k_2, \ldots, k_\ell)$, we denote the critical threshold associated with that system as $p_c(\boldsymbol{k},\boldsymbol{\alpha})$ and define the fraction of the functional giant components at the steady-state as $P_{A_\infty}(p;\boldsymbol{k},\boldsymbol{\alpha})$ and $P_{B_\infty}(p;\boldsymbol{k},\boldsymbol{\alpha})$. With this setup, we have

Proposition 5.2: For any α in $[0,1]$, we have

$$\begin{gathered} P_{A^1_\infty}(p;k) \geq P_{A_\infty}(p;\boldsymbol{k},\boldsymbol{\alpha}), \quad P_{B^1_\infty}(p;k) \geq P_{B_\infty}(p;\boldsymbol{k},\boldsymbol{\alpha}) \\ p_{c_1}(k) \leq p_c(\boldsymbol{k},\boldsymbol{\alpha}). \end{gathered}$$

where $k = \sum_{i=1}^{\ell} \alpha_i k_i$.

Proposition 5.2 can be established in the same manner with Proposition 5.1; the details are omitted due to space limitations. This result implies that the regular allocation strategy leads to a smaller p_c and a better robustness than any unequal allocation strategy. Note again that the proofs of Proposition 5.1 and Proposition 5.2 do not take any assumptions on network intra-structures, and hence the optimality of the regular allocation strategy is in place for interdependent networks with arbitrary intra-degree distributions.

VI. Conclusion and Future Work

We study a cyber-physical system in which a cyber-network overlays a physical-network. To improve network robustness against random node failures, we develop and study a regular allocation strategy that allots a fixed number of inter-network edges to each node. Our findings reveal that the proposed regular allocation strategy yields the optimal robustness amongst all strategies when no information regarding the intra-structures of the networks are available. We expect that in the presence of such information, the topology of the networks can be exploited to improve further the robustness of cyber-physical systems against cascading failures. Clearly, there are still many open questions on network interdependence in a cyber-physical system, and we are currently investigating related issues along this avenue.

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