Positive Operator Valued Measures: A General Setting for Frames

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***Abstract*— We describe the concept of a *framed positive operator valued measure* as a framework for encompassing the various generalizations of frames.**

I. INTRODUCTION

Frames have become a standard tool in signal processing, allowing uniform descriptions of many linear but non-orthogonal transform techniques as well as sensor signal processing methods. In particular they serve as models for problems in array processing, in sampling, and in spectral and other transform methods, including wavelets, for time series. Frames play an important role in compressive sensing. They arose first in the work of Duffin and Schaeffer [8], where they were used as an abstraction of sampled Fourier transforms. Little interest was shown in them until the paper [6] of Daubechies, Grossmann, and Meyer. Subsequently they were taken up by numerous authors; we refer the reader to [5], [12], [9], [13] for further details of the theory and a discussion of the many applications of frames. The standard definition of a frame is as a collection $\mathcal{F} = \{\zeta_r : r = 1, 2 \ldots, R\}$ of elements of a Hilbert space $\mathfrak{H}$, where R may be ∞, with positive constants $A \le B$ such that, for all $\xi \in \mathfrak{H}$,

$$A\|\xi\|^2 \le \sum_{r=1}^{R} |\langle \zeta_r, \xi\rangle|^2 \le B\|\xi\|^2. \tag{1}$$

Roughly speaking a projection $\xi \to \langle \xi, \zeta_r\rangle$ of a vector ξ representing the state of a system onto an individual member ζ_r of a frame may be seen as a measurement of that system, and the aim is to reconstruct the state ξ from the collection of all individual measurements $(\langle \xi, \zeta_r\rangle)_r$ in a robust way. The frame condition as stated in (1) expresses the ability to do that, and the frame bounds provide a measure of robustness. If $A = B$ the frame is said to be *tight*. Orthonormal bases are special cases of frames, and for these $A = B = 1$.

Several generalizations of the basic concept of frames have been invented; these include, in particular, the possibility that the family of vectors is indexed by a continuum, rather than integers, resulting in *generalized frames*. There are various formulations of this; see in particular [1]. From our perspective in this paper, the infrastructure of a generalized frame is a measurable function $\Phi : \Omega \to \mathfrak{H}$, Ω, where $(\Omega, \mathcal{B}, \mu)$ is a measure space (such as the real line with Lebesgue measure), which has the property that $\Phi_\xi : t \mapsto \langle \xi, \Phi(t)\rangle$ is in $L^2(\Omega, \mu, \mathfrak{H})$ for all ξ in $\mathfrak{H}$. The set Ω plays the role of an index set, and the function Φ is the replacement for a frame. The natural replacement for the frame condition (1) is,

$$A\|\xi\|^2 \le \int_\Omega |\langle \Phi(t), \xi\rangle|^2 \, d\mu(t) \le B\|\xi\|^2. \tag{2}$$

We will occasionally use the Dirac "bra-ket" notation: for vectors $\eta, \zeta \in \mathfrak{H}$, $|\eta\rangle\langle\zeta|$ denotes the (non-orthogonal projection) operator

$$|\eta\rangle\langle\zeta|(\xi) = \langle \zeta, \xi\rangle\eta, \quad (\xi \in \mathfrak{H}).$$

When $\eta = \zeta$ and $\|\eta\| = 1$, the projection $\pi_\eta = |\eta\rangle\langle\eta|$ is an orthogonal projection onto the subspace generated by η. Now (2) becomes

$$A\mathbb{I} \le \int_\Omega |\Phi(t)\rangle\langle\Phi(t)| \, d\mu(t) \le B\mathbb{I} \tag{3}$$

where the inequalities means that the differences are positive operators on the Hilbert space, and the integral is in the weak sense. Briefly, for a suitable measurable family of operators $(B(t))_{t\in\Omega}$ on the Hilbert space $\mathfrak{H}$, the integral $\int_\Omega B(t)\, d\mu(t)$ is defined to be the operator D satisfying

$$\langle \xi, D\eta\rangle = \int_\Omega \langle \xi, B(t)\eta\rangle, d\mu(t), (\xi, \eta \in \mathfrak{H}).$$

Fusion frames are another generalization of the concept of a frame but taking a different direction. These have received a lot of attention in the signal processing literature recently; see for example, [10], [3], [17], [2]. In *fusion frames* the one dimensional projections π_{ζ_r} are replaced by projections π_{W_r} onto potentially higher dimensional closed subspaces $W_r \subset \mathfrak{H}$. Thus a *fusion frame* $\mathcal{F}$ is a family $(W_r)_r$ of closed subspaces of $\mathfrak{H}$ and a family of weights (w_r), where $w_r \in \mathbb{R}^+$ with the "frame condition"

$$A\|\xi\|^2 \le \sum_{r=1}^{R} w_r^2 \|\pi_{W_r}(\xi)\|^2 \le B\|\xi\|^2.$$

Fusion frames are seen as a way of representing the problem of fusion of multiple measurements in, for example, a sensor network, where each projection corresponds to a node of the network, and the fusion frame itself, as its name implies, as the mechanism for fusion of these measurements centrally.

One can combine the two generalizations of the concept of frames described so far into a single overarching generalization which, for want of a better term, we call a *generalized fusion frame*, in which a fusion frame is replaced by a pair of measurable functions $\mathcal{F} = (\Phi, w)$. In this case $\Phi : \Omega \to \mathfrak{P}(\mathfrak{H})$, and $t \mapsto w(t) : \Omega \to \mathbb{R}^+$, where $\mathfrak{P}(\mathfrak{H})$ is the space (with the weak operator topology) of orthogonal projections (of any rank, including possibly ∞) on $\mathfrak{H}$, and Φ is *measurable* in the weak sense that $t \to \langle \eta, \Phi(t)\xi \rangle$ is measurable for each $\xi, \eta \in \mathfrak{H}$. As part of the definition, we demand too that the function $t \to \Phi(t)\xi$ is in $L^2(\Omega, \mu, \mathfrak{H})$ for all $\xi \in \mathfrak{H}$. The frame condition in operator form in (3) becomes

$$A\mathbb{I} \le \int_\Omega w(t)^2 \Phi(t)\, d\mu(t) \le B\mathbb{I}.$$

in this context, where again $0 < A \le B$.

As we shall describe, the definition of a generalized fusion frame turns out to be little different from a concept that has been prevalent in the quantum physics literature for many years; that of a positive operator valued measure (POVM). This is not surprising from a signal processing viewpoint, as the concept of POVM was introduced and developed in quantum mechanics as a means to represent the most general form of quantum measurement of a system. Here we discuss a generalization of the POVM concept as used in quantum mechanics, which enables us to encompass the theory of fusion frames, and even generalized fusion frames as well, of course, as generalized frames. Our point is that once generalized fusion frames are accepted there is no reason in not going the extra mile and setting the discourse in term of POVMs. The latter enable the importation of much theory from the quantum mechanics literature as well as decompositions that are not so apparent from the frame formalism.

A key result here is the classical theorem of Naimark [14] which, long before frames became popular in signal processing, or POVMs were used in quantum mechanics, formalized analysis and synthesis in this general context and, when applied to the cases above, exactly reproduces those notions.

We will describe positive operator valued measures, the theorem of Naimark, and how these relate to frames and their generalizations. In this brief description of the relationship between POVMs and the generalizations of frames, it will only be possible to touch on the power of the POVM formalism.

II. Analysis and Synthesis

The various concepts of frame, fusion frame, and generalized frame all give rise to analysis and synthesis operations, which break up any *signal* vector $\xi \in \mathfrak{H}$ into its "frame" components and reconstruct them. In the case of a frame, the analysis operator F takes "signals" in the Hilbert space $\mathfrak{H}$ to coefficients in the space ℓ^2 of square-summable sequences; that is, F is given by $F(\xi) = (\langle \xi, \zeta_r \rangle)_{1 \le r \le R}$. That this is in ℓ^2 is guaranteed by the frame bounds. The adjoint map $F^* : \ell^2 \to \mathfrak{H}$ is

$$F^*((a_r)) = \sum_r a_r \zeta_r,$$

and corresponds to synthesis of a signal from the frame coefficients. The *frame operator* is F^*F and it satisfies

$$A\mathbb{I}_{\mathfrak{H}} \le F^*F \le B\mathbb{I}_{\mathfrak{H}}. \tag{4}$$

In the case of fusion frames, definition of the analysis operator requires the replacement of the analysis space ℓ^2 of a frame by the natural generalization, the Hilbert space direct sum $\bigoplus_{r=1}^{R} W_r$, consisting of all sequences $(\xi_r)_r$, where $\xi_r \in W_r$ and $\sum_{r=1}^{R} \|\xi_r\|^2 < \infty$. Then the analysis operator $F : \mathfrak{H} \to \bigoplus_{r=1}^{R} W_r$ and is given by

$$F(\xi) = (w_r \pi_{W_r}(\xi)) \in \bigoplus_{r=1}^{R} W_r, \quad (\xi \in \mathfrak{H}).$$

The adjoint map F^*, is given by

$$F^*((\xi_r)_r) = \sum_{r=1}^{R} w_r \xi_r \in \mathfrak{H}, \quad ((\xi_r)_r \in \bigoplus_{r=1}^{R} W_r.$$

The frame bound conditions guarantee that everything is well-defined. In this case, the *fusion frame operator* is given by

$$F^*F(\xi) = \sum_{r=1}^{R} w_r^2 \pi_{W_r}(\xi),$$

and the same kind of frame bound inequality as in (4) exists for fusion frames.

For the generalized frame described in the Introduction, the frame operator $F : \mathfrak{H} \to L^2(\Omega, \mu)$ is given by

$$F(\xi)(t) = \langle \xi, \Phi(t) \rangle, \quad (t \in \Omega,\ \xi \in \mathfrak{H}),$$

and its adjoint by

$$F^*(f) = \int_\Omega f(t)\Phi(t)\, d\mu(t) \in \mathfrak{H}, \quad (f \in L^2(\Omega, \mu)).$$

Again F^*F satisfies the inequalities (4).

For generalized fusion frames there is a corresponding definition of analysis and synthesis operators but its construction requires the definition of direct integrals of Hilbert spaces [7], and in any case the ideas will be subsumed under our more general development below.

We observe, trivially that, in each case, the synthesis operator does not reconstruct the analysed signal; that is $F^*F \ne \mathbb{I}$ in general. In the case of a frame, the inversion of the analysis operator is performed by invoking a *dual frame*. There are various different usages of this terminology in the literature (see [3], [11], [13]) . For our purposes, given a frame (ζ_r) on the Hilbert space $\mathfrak{H}$, a *dual frame* $(\widetilde{\zeta_r})$ satisfies

$$\xi = \sum_{r=1}^{R} \langle \xi, \zeta_r \rangle \widetilde{\zeta_r} = \sum_{r=1}^{R} \langle \xi, \widetilde{\zeta_r} \rangle \zeta_r. \tag{5}$$

In other words, the dual frame inverts the analysis and synthesis operations of the original frame to give perfect recontruction. Such a dual frame always exists; indeed, it is easy to check that

$$\widetilde{\zeta_r} = (F^*F)^{-1}(\zeta_r) \tag{6}$$

has the appropriate property. Dual frames as defined in (5) are not in general unique; the one in equation (6) is called the *canonical dual frame*. In the case of a fusion frame $(W_r, w_r)_r$, there also exist dual fusion frames; the canonical dual fusion frame being $((F^*F)^{-1}(W_r), w_r)$; see [11] for proofs of their existence and properties.

III. Positive Operator Valued Measures

At this point we introduce the key definition of a *framed POVM* as a function on the σ-algebra of Borel sets $\mathfrak{B}(\Omega)$ of a space Ω, and taking values in the space of positive operators $\mathfrak{B}^+(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$. While avoiding most of the mathematical technicalities, we assume that Ω is a "nice" space such as a complete separable metric space or a locally compact second countable space, in order that the concept of regularity of measures is meaningful and useful, though in this brief account we will not discuss regularity. A *framed POVM* is such a function $M : \mathfrak{B}(\Omega) \to \mathfrak{B}^+(\mathfrak{H})$ satisfying the following two conditions:

POVM-1) For all ξ in $\mathfrak{H}$, $\omega \to \langle \xi, M(\omega)\xi \rangle$ is a regular Borel measure on $\mathfrak{B}(\Omega)$, denoted by μ_ξ and

POVM-2) $A\mathbb{I} \leq M(\Omega) \leq B\mathbb{I}$ for some $0 < A \leq B < \infty$.

The numbers A, B are called the *frame bounds* as in the case of frames. Without the condition POVM-2), we refer to the object as a POVM without the epithet "framed". Such an object is a measure with values in the set of positive operators on a Hilbert space $\mathfrak{H}$, though the *countable* aspect of its additivity is only in a weak sense. In the quantum context POVM-2) is replaced by the more strict requirement that $M(\Omega) = I$. A framed POVM is *tight* if $A = B$, and if $A = B = 1$, we call M a *probability POVM*; the latter are used in quantum mechanics as the most general form of quantum measurement.

As an example of a framed POVM, consider first a fusion frame (w_r, W_r) on $\mathfrak{H}$; define Ω to be the set of integers from 1 to R with the σ-field taken to be the collection of all subsets of Ω. Now

$$M(\omega) = \sum_{r \in \omega} w_r \pi_{W_r}. \tag{7}$$

It is straightforward to see that this satisfies both parts of the definition of the definition of a framed POVM with the frame bounds being the bounds in the definition of a fusion frame. Thus every fusion frame, and hence every frame, is trivially represented as a framed POVM.

If $\mathcal{F} = \Phi : \Omega \to \mathfrak{H}$ is a generalized frame, then we can define a POVM by

$$M(\omega) = \int_\omega |\Phi(r)\rangle\langle\phi(r)|\, d\mu(r), \tag{8}$$

and again this is a framed POVM with the same frame bounds as those of the generalized frame.

The aim of framed POVMs in this context is to provide a general framework for analysis and reconstruction of signals. We shall consider how to define analysis and synthesis in a fairly general context over the next section.

Why generalize to framed POVMs? The generality of framed POVMS provides both the freedom to develop new constructs and the prior theory developed in the physics literature for these objects. We believe that it provides a powerful machine for investigating ideas of analysis and synthesis in signal processing. Moreover, it turns out that framed POVMs are barely more general than generalized fusion frames we briefly discussed in the Introduction. We illustrate these concepts here but reserve much of the theory for a later paper.

IV. Spectral Measures and the Naimark Theorem

A very special kind of POVM S is a *spectral* POVM; these satisfy the additional constraint that

$$S(\omega_1 \cap \omega_2) = S(\omega_1)S(\omega_2), \qquad (\omega_1, \omega_2 \in \mathcal{B}),$$

and arise, for example, in the Spectral Theorem of a Hermitian operator Q on a Hilbert space (see for example [18]).

While the definition of a spectral POVM does not force it to be a probability POVM, it does force $S(\Omega)$ to be a projection, and all $S(\omega)$ to be projections dominated by $S(\Omega)$, that is

$$S(\omega)S(\Omega) = S(\Omega)S(\omega) = S(\omega) \quad (\omega \in \mathcal{B});$$

Thus, all of the action takes place in a closed subspace $S(\Omega)\mathfrak{H}$ of $\mathfrak{H}$, and there is usually no loss in assuming that $S(\Omega)\mathfrak{H} = \mathfrak{H}$. Indeed this is forced if, in addition, we require the spectral POVM to be framed. Accordingly we will assume that $S(\Omega) = \mathbb{I}_\mathfrak{H}$ whenever discussing a spectral POVM in this paper. This in turn, of course, forces the POVM to be framed; in fact a probability POVM. As an intuitive guide, spectral POVMs play the same role relative to framed POVMs as orthogonal bases do relative to frames.; they generalize orthogonal bases just as framed POVMs generalize frames.

Now we have the machinery to state the key theorem on POVMs due to Naimark [14], who stated it for POVMs without the framed condition; the following version is a relatively straightforward adaptation to our context.

Theorem 1 *Let $(\Omega, M, \mathfrak{H})$ be a framed POVM with frame bounds A and B. Then there is an "auxiliary" Hilbert space $\mathfrak{H}_\sharp$, a spectral POVM S with values in $\mathfrak{P}(\mathfrak{H}_\sharp)$, and a bounded linear map $V : \mathfrak{H}_\sharp \to \mathfrak{H}$, such that*

$$M(\omega) = VS(\omega)V^* \qquad (\omega \in \mathcal{B}).$$

and $A\mathbb{I} \leq VV^* \leq B\mathbb{I}$.

We refer to the collection $(\Omega, S, \mathfrak{H}_\sharp, V)$ as a *Naimark representation* of the framed POVM M.

A Naimark representation is *minimal* if the set

$$\{S(\omega)V^*\xi : \xi \in \mathfrak{H},\ \omega \in \mathcal{B}\}$$

is dense in $\mathfrak{H}_\sharp$. Minimal Naimark representations are essentially unique in the sense that if $(\Omega, S, \mathfrak{H}_\sharp, V)$ and $(\Omega, S', \mathfrak{H}'_\sharp, V')$ are two such representations then there is a surjective isometry $T : \mathfrak{H}_\sharp \to \mathfrak{H}'_\sharp$ such that $V'T = V$ and $T^{-1}S'(\omega)T = S(\omega)$ for all $\omega \in \mathcal{B}$. At this point we remark that the fashionable way (see [15]) to handle the Naimark representation is to convert POVMs to (completely) positive operators on commutative C^*-algebras via integration and then Naimark's theorem becomes a special case of Stinespring's

Theorem [16], which does not require commutativity of the C^*-algebra. A description of this approach would significantly lengthen the paper and we omit it in this brief introduction.

Example 1 A generalized frame $\Phi : \Omega \to \mathfrak{H}$ on the measure space (Ω, μ) and with frame bounds $0 < A \leq B$, gives rise to a framed POVM as in (8). To form a Naimark representation, let the Hilbert space $\mathfrak{H}_\sharp$ to be $L^2(\Omega, \mu)$, and let the spectral measure S be the canonical one on this space, namely,

$$S(\omega)f(t) = \mathbb{I}_\omega(t)f(t) \qquad (f \in L^2(\Omega, \mu)).$$

This is clearly a spectral measure since $\mathbb{I}_\omega.\mathbb{I}_{\omega'} = \mathbb{I}_{\omega\cap\omega'}$. The map $V : L^2(\Omega, \mu) \to \mathfrak{H}$ is defined by

$$V(f) = \int_\Omega f(t)\Phi(t)\, d\mu(t)$$

where $f(t)\Phi(t)$ is just the product of the scalar $f(t)$ with $\Phi(t) \in \mathfrak{H}$. It can be checked that this is indeed a (the) minimal Naimark representation of the framed POVM in (8).

Example 2 A fusion frame given, say, by (w_r, W_r), where $(w_r)_r$ is a sequence of (non-negative) weights, and W_r are subspaces of the Hilbert space $\mathfrak{H}$ which together satisfy (1), produces a framed POVM as in (7). In this case, we take $\mathfrak{H}_\sharp$ to be the formal direct sum $\bigoplus_{r=1}^{R} W_r$, that is, all sequences $(\xi_r)_r$ where $\xi_r \in W_r$ and $\sum_{r=1}^{R} \|\xi_r\|^2 < \infty$. The appropriate spectral measure S is defined on $\mathbb{N}$, by

$$S(K) = \bigoplus_r r = 1^R \pi_{W_r}^{\mathfrak{H}_\sharp}, \tag{9}$$

and $\pi_{W_r}^{\mathfrak{H}_\sharp}$ is the projection onto W_r in $\mathfrak{H}_\sharp$. The map $V : \mathfrak{H}_\sharp \to \mathfrak{H}$ is given by

$$V(\xi_r) = \sum_{r=1}^{R} w_r \xi_r, \tag{10}$$

where the summands are now considered as members of $\mathfrak{H}$. The summability of the weights guarantees that the sum in (10) converges in the Hilbert space $\mathfrak{H}$, because

$$\sum_{r=1}^{R} \|w_r \xi_r\| \leq \Big(\sum_{r=1}^{R} w_r^2\Big)^{1/2} \Big(\sum_{r=1}^{R} \|\xi_r\|^2\Big)^{1/2}, \tag{11}$$

by Cauchy-Schwarz. Thus, V is a linear map from $\mathfrak{H}_\sharp \to \mathfrak{H}$ which is bounded; in fact, by (11),

$$\|V\| \leq \Big(\sum_{r=1}^{R} w_r^2\Big)^{1/2}.$$

The adjoint linear map $V^* : \mathfrak{H} \to \mathfrak{H}_\sharp$ is given by

$$V^*(\xi) = (w_r \pi_r(\xi))_r.$$

Observe that with $\omega = \Omega$, we obtain

$$M(\Omega) = VS(\Omega)V^* = VV^*,$$

since $S(\Omega) = \mathbb{I}$. The frame bounds then say that the positive definite matrix VV^* has spectrum lying in the interval $[A, B]$. If the fusion frame is tight then $VV^* = A\mathbb{I}$.

Comparing the descriptions in Section II with the examples given here, we see that Naimark's Theorem provides exactly the machinery for discussing analysis and synthesis operators in a general context, and we do that next.

V. Analysis and Synthesis for General POVMs

The preceding examples show that the Naimark representation provides the mechanism for analysis and synthesis in POVMs that precisely extends that of frames and fusion frames. To be specific, let $(\Omega, M, \mathfrak{H})$ be a POVM, though this stage we are not concerned about it having frame bounds. Let $(\Omega, S, V, \mathfrak{H}_\sharp)$ be the corresponding mimimal Naimark representation. Then the *analysis space* is $\mathfrak{H}_\sharp$, the *analysis operator* $V^* : \mathfrak{H} \to \mathfrak{H}_\sharp$, and the *synthesis operator* $V : \mathfrak{H}_\sharp \to \mathfrak{H}$. The Naimark representation also presents us with a gadget, namely the spectral measure S for keeping track of the labelling of the frame, fusion frame. and generalized fusion frame.

Thus the *analysis* of an element ξ of $\mathfrak{H}$ is the Hilbert space-valued (more precisely $\mathfrak{H}_\sharp$-valued) measure)

$$\mathcal{A}(\xi) = \widehat{\xi} : \omega \mapsto S(\omega)V^*\xi : \Omega \to \mathfrak{H}_\sharp. \tag{12}$$

In the case of, say, a frame, this measure (now on $\mathbb{N}$) is the one that assigns a frame coefficient to each integer. Given a measure $\rho : \Omega \to \mathfrak{H}_\sharp$ as in (12), its integral (the *synthesis*) is

$$\mathcal{S}(\rho) = V \int_\Omega d\rho(t). \tag{13}$$

As the examples in the preceding section show, these analyses and syntheses operators correspond precisely to those of classical frames, fusion frames, and generalized fusion frames.

VI. Canonical Representations and POVMs

As an illustration of the kind of theory available once we adopt the POVM formalism, here we put together the Naimark theorem with the canonical representation of spectral POVMs (see for example, [18]) to provide a kind of canonical description of a POVM. This canonical representation decomposes $\mathfrak{H}_\sharp$, the analysis space, of the POVM $(M, \Omega, \mathfrak{H})$ into a direct sum of $\bigoplus_n \mathfrak{K}_n$ such that

1) each of the spaces $\mathfrak{K}_n$ is invariant under the spectral measure:
$$S(\omega)\mathfrak{K}_n \subset \mathfrak{K}_n \qquad (1w \in \mathcal{B}, n \in \mathbb{N}^*)$$
and
2) the restriction of S to $\mathfrak{K}_n$ has uniform multiplicity; in other words, $\mathfrak{K}_n \simeq \mathbb{C}^{u_n} \otimes L^2(\mu_n)$ if $\mathfrak{K}_n$ has finite multiplicity u_n, or $\mathfrak{K}_n \simeq \ell^2 \otimes L^2(\mu_n)$ if the multiplicity is infinite.

This representation is essentially unique up to unitary equivalence and replacement of each of the measures μ_n by one equivalent to it (that is, having the same null sets). Let P_n

denote the projection onto $\mathfrak{K}_n$ regarded as a subspace of $\mathfrak{K}$. Under the (minimal) Naimark representation, $V : \mathfrak{H}_\sharp \to \mathfrak{H}$ is such that $V^*S(\omega)V = M(\omega)$ for $\omega \in \mathcal{B}$. We can decompose V as $V = \sum_n VP_n = \sum_n V_n$. The image of V_n^* is in $\mathfrak{K}_n$ so that,

$$M(\omega) = \sum_n V_n S(\omega) V_n^* \qquad (\omega \in \mathcal{B}).$$

The map $\omega :\mapsto V_n S(\omega) V_n^*$ is a POVM with values in $\mathfrak{K}_n$ which we denote by M_n; or more precisely, $(\Omega, M_n, \mathfrak{H}_n)$, where $\mathfrak{H}_n$ is the closure of the image of V_n . The maps $M_n(\omega) : \omega \mapsto V_n^* S\omega)V_n$ are possibly not framed POVMs. However,

$$M_n(\Omega) = V_n V_n^*$$

and so if M is a framed POVM with frame bounds $0 < A < B$,

$$A\mathbb{I}_{\mathfrak{H}} \leq \sum M_n(\Omega) = \sum_n V_n V_n^* \leq B\mathbb{I}_{\mathfrak{H}}.$$

Observe that $V_n^* V_n V_m^* V_m = 0$ for $n \neq m$, since the image of V_m^* lies in $\mathfrak{K}_m$ which is in the kernel of V_n. In an obvious sense,

$$M = \sum_n M_n.$$

Thus every framed POVM is a sum of "uniform multiplicity" POVMs, though these are no longer framed.

Example 3 Consider a "classical" frame $\mathcal{F} = \{\xi_k\}_k$ in a Hilbert space $\mathfrak{H}$ with frame bounds $0 < A \leq B$, and its corresponding framed POVM M In this case $\mathfrak{H}_\sharp$ is ℓ^2, $V : \mathfrak{H}_\sharp \to \mathfrak{H}$ is given by $V(\mathbf{e}_k) = \xi_k$ where $(\mathbf{e}_k)$ is the standard basis of ℓ^2, and the spectral measure is given by

$$S(K) = \sum_{k \in K} |\mathbf{e}_k\rangle\langle\mathbf{e}_k| \qquad (K \subset \mathbb{N}).$$

Alternatively this can be redefined by regarding members of ℓ^2 as functions on $\mathbb{N} = \Omega$ by $S(K)(f) = \mathbb{I}_K f$, so that the spectral measure is uniform with multiplicity 1.

Example 4 The case of a fusion frame specified by (w_r, W_r), where $(w_r)_r$ is a sequence of (non-negative) weights, and W_r are subspaces of the Hilbert space $\mathfrak{H}$ is more complicated than the frame case. In this case, the spectral measure is described earlier (see (9)). For each member m of $\mathbb{N}^*$ let

$$U_m = \{r \in \mathbb{N} : \dim W_r = m\}, \qquad (m \in \mathbb{N}^*).$$

Then

$$Y_m = \bigoplus_{r \in U_m} W_r \subset \mathfrak{H}_\sharp$$

is equal to $\mathbb{C}^m \otimes \ell^2(Y_m)$ or if $m = \infty$ $\ell^2 \otimes \ell^2(Y_m)$, where $\ell^2(B)$ for $B \subset \mathbb{N}$ is

$$\{(a_r)_{r \in B} : \sum_{r \in B} |a_r|^2 < \infty\}.$$

Evidently, Y_m has uniform multiplicity m, and the measure μ_m is the counting measure on U_m, though of course some of the U_ms could be empty. If all of the W_r have the same dimension then the spectral measure has uniform multiplicity.

VII. Dual POVMs

As we have seen in Section II each of the various versions of a frame and its generalizations gives rise to a "dual" concept with the definition of a canonical dual frame. This is also possible for framed POVMs, and indeed is relatively straightforward once the Naimark representation is available. We assume a POVM $(\Omega, M, \mathfrak{H})$ and its minimal Naimark representation $(\Omega, S, \mathfrak{H}_\sharp, V)$. Now we can define the *canonical dual POVM* to be that with Naimark representation $(\Omega, S, \mathfrak{H}_\sharp, (VV^*)^{-1}V)$. In other words the canonical dual POVM $(\Omega, \widetilde{M}, \mathfrak{H})$ is given by the equation

$$\widetilde{M}(\omega) = (VV^*)^{-1} VS(\omega)V^*(VV^*)^{-1}.$$

The frame condition on M guarantees the existence of the inverse operators $(VV^*)^{-1}$ (see Theorem 1). Now we observe that

$$\begin{aligned} M(\omega)\widetilde{M}(\omega) &= \big(VS(\omega)V^*\big)\big((VV^*)^{-1}VS(\omega)V^*(VV^*)^{-1}\big) \\ &= VS(\omega)V^*(VV^*)^{-1} \\ \widetilde{M}(\omega)M(\omega) &= \big((VV^*)^{-1}VS(\omega)V^*(VV^*)^{-1}\big)\big(VS(\omega)V^*\big) \\ &= (VV^*)^{-1}VS(\omega)V^*. \end{aligned}$$

In particular, since $S(\Omega) = \mathbb{I}$,

$$M(\Omega)\widetilde{M}(\Omega) = \widetilde{M}(\Omega)M(\Omega) = \mathbb{I}.$$

From the point of view of analysis and synthesis, if $\xi \in \mathfrak{H}$, its analysis with respect to M is as described in (12). If we now apply synthesis according to the canonical dual POVM $\widetilde{M}$ and using equation (13) with appropriate modifications,

$$\begin{aligned} \widetilde{\mathcal{S}}(\mathcal{A}(\xi)) &= (VV^*)^{-1}V \int_\Omega d\rho = (VV^*)^{-1}V \int_\Omega dS(\omega)V^*\xi \\ &= (VV^*)^{-1}VS(\Omega)V^*\xi = \xi. \end{aligned} \quad (14)$$

A similar formula is available for $\mathcal{S}(\widetilde{\mathcal{A}}(\xi))$.

VIII. Radon-Nikodym Theorem for POVMs

Here we describe how, at least in the case of finite dimensional Hilbert spaces, where many of the applications of frame ideas are, the concept of a framed POVM is essentially the same as that of a generalized fusion frame as described in the Introduction. Let $(\Omega, M, \mathfrak{H})$ be a framed POVM. We shall make the additional assumption throughout this section that $\mathfrak{H}$ is finite-dimensional. Of course, in most signal processing applications this is not an issue, and there are many instances in which it can be weakened. The reason for this assumption is that we need to define a real-valued Borel measure: $\mu(\omega) = \mathrm{Tr}(M(\omega))$ for $\omega \in \mathcal{B}$. This is a positive (real-valued) regular Borel measure on Ω. This measure plays a key role in the following result, which is called the *Radon-Nikodym* Theorem for POVMs (see [4]).

Theorem 2 *Let $(\Omega, M, \mathfrak{H})$ be a POVM with $\mathfrak{H}$ finite dimensional. Then there is a regular positive real-valued measure*

μ on $(\Omega, \mathcal{B})$ and an operator-valued bounded measurable function $F : \Omega \to \mathfrak{B}^+(\mathfrak{H})$ such that

$$M(\omega) = \int_\omega F(t)\, d\mu(t) \qquad (\omega \in \mathcal{B}).$$

The measure μ is called the *base measure* of the POVM and f the *Radon-Nykodym derivative* of the POVM. This representation is very useful in facilitating constructions on POVMs on finite dimensional Hilbert spaces.

Corollary 1 *If, in addition, M is a framed POVM with frame bounds $0 < A \leq B$, then*

$$A\mathbb{I}_\mathfrak{H} \leq \int_\Omega F(t)\, d\mu(t) \leq B\mathbb{I}_\mathfrak{H}$$

It is again useful and instructive to consider how the Radon-Nykodym Theorem works for the motivating examples. Most importantly, this result shows that, with the finite dimensional assumption, framed POVMs correspond exactly to generalized fusion frames as we have described them in the Introduction.

Example 5 Let $\mathcal{F} = (\xi_n)_{n \in W}$ be a frame in a finite dimensional Hilbert space, where W is some subset of the set of positive integers. The associated POVM is given by $M(A) = \sum_{n \in A} |\xi_n\rangle\langle\xi_n|$ for subsets A of W. In this case, $\Omega = W$, the measure μ is counting measure on W, and the operator valued function F is given by

$$F(n) = |\xi_n\rangle\langle\xi_n| \qquad (n \in W).$$

Note that in this case there is no need for the finite dimensional restriction on $\mathfrak{H}$. A POVM constructed from a frame in this way automatically satisfies a Radon-Nykodym theorem with respect to the counting measure.

Example 6 In the case of a generalized frame, $\mathcal{F} = \{\Phi : \Omega \to \mathfrak{H}\}$, where again $\mathfrak{H}$ is a finite dimensional Hilbert space, the POVM is

$$M(\omega) = \int_\omega |\Phi(t)\rangle\langle\Phi(t)|\, d\mu(t), \qquad (15)$$

so that the operator valued function is exactly

$$F(t) = |\Phi(t)\rangle\langle\Phi(t)|$$

As in the previous case, a POVM constructed in this way, even when $\mathfrak{H}$ is not finite dimensional, automatically satisfies a Radon-Nikodym theorem with respect to the measure μ — of course the measure is supplied with the definition.

Example 7 For a fusion frame $(W_r, w_r)_r$ the Radon-Nikodym decomposition is straightforward. The space Ω is $\mathbb{N}$, and μ is the counting measure. The function $F : \mathbb{N} \to \mathfrak{B}^+(\mathfrak{H})$ is $F(r) = w_r^2 \pi_{W_r}$, $(r \in \mathbb{N})$. This coincides with our previous observation that the POVM in this case is defined by

$$M(\omega) = \sum_{r \in \omega} w_r^2 \pi_{W_r} \qquad (\omega \subset \mathbb{N}).$$

Note that, even though the values are not projections, they are scalar multiples of projections. We could replace the counting measure μ by $\nu(r) = w_r^2$ for $r \in \mathbb{N}$, and the POVM formula (15) would become

$$M(\omega) = \int_\mathbb{N} \pi_{W_r}\, d\nu(r) \qquad (\omega \subset \mathbb{N}),$$

so that in this case the values of F are true projections.

We say that a POVM $(\Omega, M, \mathfrak{H})$ is *decomposable* if there is an essentially bounded measurable function $F : \Omega \to \mathfrak{B}^+(\Omega)$ and a measure μ on Ω such that

$$M(\omega) = \int_\omega F(t)\, d\mu(t) \qquad (\omega \in \mathcal{B}).$$

We have seen that if $\dim \mathfrak{H} < \infty$ then the POVM is decomposable, and that every POVM arising from a (generalized) frame, even when $\dim \mathfrak{H}$ is not finite dimensional.

In effect decomposable framed POVMs correspond exactly to the definition given of generalized fusion frames to capture both of the generalizations of frames to, on the one hand, fusion frames, and on the other hand generalized frames. A decomposable framed POVM is the right extension of both fusion frames and generalized frames.

REFERENCES

[1] S. T. Ali, J. P. Antoine, and J. P. Gazeau. Continuous frames in Hilbert space. *Annals of Physics*, 222(1):1 – 37, 1993.

[2] M. S. Asgari and A. Khosravi. Frames and bases of subspaces in Hilbert spaces. *J. Math. Anal. Appl.*, 308:541–553, 2005.

[3] P. G. Casazza and G. Kutyniok. Frames of subspaces. In *Wavelets, Frames and Operator Theory*, volume 345 of *Contemp. Math*, pages 87–113, 2004.

[4] Giulio Chiribella, Giacomo Mauro D'Ariano, and Dirk Schlingemann. Barycentric decomposition of quantum measurements in finite dimensions. *J. Math. Physics*, 51:022111, 2010.

[5] Ole Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2003.

[6] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27:1271–1283, 1986.

[7] J. Dixmier. *Von Neumann Algebras*. North-Holland, Amsterdam, 1981.

[8] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic fourier series. *Transactions of The American Mathematical Society*, 72:341–341, 1952.

[9] H.G. Feichtinger and T. Strohmer, editors. *Gabor Analysis and Algorithms: Theory and Applications*. Birkhäuser, Boston, MA, 1998.

[10] M. Fornasier. Quasi-orthogonal decompositions of structured frames. *J. Math. Anal. Appl.*, 289:180–199, 2004.

[11] P. Gavruta. On the duality of fusion frames. *Journal of Mathematical Analysis and Applications*, 333:871–879, September 2007.

[12] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, MA, 2001.

[13] C. Heil and D. Walnut. Continuous and discrete wavelet transforms. *SIAM Rev.*, 31:628–666, 1989.

[14] M.A. Naimark. On a representation of additive operator set functions. *Dokl.Acad.Sci. SSSR*, 41:373–375, 1943.

[15] K.R. Parthasarathy. Extremal decision rules in quantum hypothesis testing. In *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, volume 2, pages 557–568. World Scientific, 1999.

[16] W. Forrest Stinespring. Positive functions on C^*-algebras. *Proc. American Math Soc.*, 6:211–216, 1955.

[17] W. Sun. G-frames and G-Riesz bases. *J. Math. Anal. Appl.*, 322:437–452, 2006.

[18] V. S. Sunder. *Functional Analysis: Spectral Theory. (English)*. Birkhäuser, 1997.