OPTIMAL WAVEFORM DESIGN FOR RADAR IN THE PRESENCE OF DOPPLER

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ABSTRACT

Optimal radar waveform design for target detection has been addressed in prior research literature under various assumptions regarding noise and clutter. A common model of the radar scene in work of this kind is a linear time-invariant operator with additive Gaussian noise that acts on the transmitted signal to produce the received signal. This model is intrinsically ill-suited to dynamic scenes or moving radar platforms because it cannot account for Doppler. This paper introduces scene models based on Hilbert-Schmidt class (HS) operators on the space of finite-energy signals. This category of models generalizes the LTI category in the sense that every LTI operator is also a HS operator, but the HS class includes operators that account for frequency shifts as well as time shifts and are thus suitable for modeling radar scenes involving Doppler. Every HS operator is uniquely expressible as a superposition of elementary time and frequency shift operators, thus providing a convenient interpretation of a scene in terms of these physically meaningful operations on the transmitted signal. Application of this perspective to waveform design for target detection in noise and to optimal receiver processing for a given waveform for target detection in clutter and noise are demonstrated.

1. INTRODUCTION

The idea of regarding a radar scene as a linear or nonlinear operator that transforms transmitted waveforms to received signals has been used in several radar contexts. In particular, this perspective has been applied to optimal waveform design for detection in noise and clutter under various resource constraints (e.g., [5, 6]). Often, the scene is modeled as a linear time-invariant (LTI) system. This class of systems is very well understood in signal processing and is of value as a radar scene model for static scenarios, but it is inadequate for situations involving Doppler. Specifically, since sinusoids are eigenfunctions of LTI systems, this mathematical model offers no means for the scene to impart a Doppler shift to a narrowband transmitted waveform.

The primary goals of this paper are to examine a broader class of linear operators, the Hilbert-Schmidt class, in the role of radar scene models and to illustrate how they can be used in the design of waveforms and optimal receiver processing for target detection in noise and clutter.

2. HILBERT-SCHMIDT OPERATORS

Denote by $L^2(\mathbb{R})$ the Hilbert space of complex signals $f : \mathbb{R} \to \mathbb{C}$ with inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t) g^*(t)\, dt$$

and norm

$$||f|| = (\langle f, f\rangle)^{1/2} = \left(\int_{\mathbb{R}} |f(t)|^2\, dt\right)^{1/2}$$

A LTI system is a linear operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that can be expressed as convolution with a fixed $h \in L^2(\mathbb{R})$; i.e.,

$$Sf(t) = [f * h](t) = \int_{\mathbb{R}} f(\tau) h(t-\tau)\, d\tau$$

Under fairly general conditions, limiting arguments permit extension of the domain of S to allow its application to ideal (Dirac) impulses $f(t) = \delta(t - t_0)$ and complex sinusoids $f(t) = e^{i\Omega t}$, even though these are not elements in $L^2(\mathbb{R})$. The function h is known as the impulse response of the system because $h = S\delta$. If f is a complex sinusoid with frequency Ω,

$$\begin{aligned} Sf(t) &= \int e^{i\omega\tau} h(t-\tau)\, dt \\ &= \int_{\mathbb{R}} e^{i\Omega t} \int_{\mathbb{R}} h(u) e^{-i\Omega u}\, du \\ &= \hat{h}(\Omega) f(t) \end{aligned}$$

The response of the system S to a sinusoidal input at a given frequency is thus a sinusoid at the same frequency whose amplitude and phase are determined by the complex constant

$$\hat{h}(\Omega) = \int_{\mathbb{R}} h(u) e^{-i\Omega u}\, du$$

(i.e., the Fourier transform of h evaluated at frequency Ω), which is known as the frequency response of the system at this frequency. The point in reiterating these well known properties of LTI systems here is to emphasize that such systems cannot manifest a shift in the frequencies present in their input signals. They are thus inadequate for modeling dynamic radar scenes involving Doppler.

The Hilbert-Schmidt (H-S) class [2] is a larger collection of linear operators on $L^2(\mathbb{R})$ that can accommodate Doppler shifts. Consider the displacement operator $D(\alpha, \tau) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$D(\alpha, \tau) f(t) = e^{i\alpha t} f(t - \tau)$$

i.e., the operator $D(\alpha, \tau)$ shifts the signal f by τ in time and by α in frequency. A Hilbert-Schmidt class operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is one that can be written as a superposition of displacement operators [4]

$$S = \int_{\mathbb{R}} \int_{\mathbb{R}} s(\alpha, \tau) D(\alpha, \tau)\, d\alpha\, d\tau \qquad (1)$$

where the "coefficients" $s(\alpha, \tau)$ form a square-summable function of the variables α and τ; i.e.,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |s(\alpha, \tau)|^2\, d\alpha\, d\tau < \infty$$

The more usual characterization of the H-S class is as the linear operators on $L^2(\mathbb{R})$ satisfying $\mathrm{Tr}(SS^\dagger) < \infty$. In terms of the expansion (1)

$$\mathrm{Tr}(SS^\dagger) = \int_{\mathbb{R}} \int_{\mathbb{R}} |s(\alpha, \tau)|^2\, d\alpha\, d\tau < \infty$$

The relation (1) can be seen as an invertible transformation between $L^2(\mathbb{R}^2)$ (i.e., the space of square-summable functions of the variables α and τ) and the H-S class. The inverse transform is

$$s(\alpha, \tau) = \mathrm{Tr}(SD(\alpha, \tau)^\dagger)$$

Note that restricting (1) to $\alpha = 0$ and writing $h(\tau) = s(0, \tau)$ gives

$$Sf(t) = \int_{\mathbb{R}} h(\tau) f(t - \tau)\, d\tau = [f * h](t)$$

So LTI operators are a subclass of the H-S class.

2.1. Scene Modeling

Consider the monostatic radar situation in which a target consists of a discrete collection of point scatterers with unknown position and unknown Doppler. Invoking narrowband and far-field assumptions [7], the H-S operator description of such a scene is

$$S = \sum_{(\alpha,\tau)} s_t(\alpha, \tau) D(\alpha, \tau) \qquad (2)$$

where the sum is taken over all pairs of Doppler shift α and time delay τ corresponding to a constituent point scatterer in the target and the complex weight $s_t(\alpha, \tau)$ represents the strength of the point scatterer at delay-Doppler position (α, τ). If the scatterers comprising the target are clustered around a nominal "target center" (α_0, τ_0) in time delay and Doppler, it is possible to re-write (2) in the form

$$S = \left[\sum_{(\alpha,\tau) \in N(0,0)} s_t(\alpha, \tau) D(\alpha, \tau) \right] D(\alpha_0, \tau_0)$$

In this expression, $N(0, 0)$ is a neighborhood of $(0, 0)$ in (α, τ) coordinates measured from the target center. The contributions of the scatters within this neighborhood are summed, each with a displacement operator $D(\alpha, \tau)$ capturing its differential range and Doppler relative to the target center. The operator $D(\alpha_0, \tau_0)$ shifts the response of this entire cluster of point scatterers by α_0 in Doppler and by τ_0 in time, corresponding to the range and radial velocity of the target center relative to the radar. This model extends to multiple targets by direct superposition.

Similarly, clutter arising from collections of point scatterers may be expressed in the form

$$C = \sum_{(\alpha,\tau)} s_c(\alpha, \tau) D(\alpha, \tau)$$

The nature of such clutter may be modeled through assumptions on the weight function $s_c : \mathbb{R}^2 \to \mathbb{C}$; e.g., deterministic, Gaussian or Markov random field, or other statistical characterizations. In the stochastic case there are advantages to assuming a continuum model for the clutter

$$C = \int_{\mathbb{R}} \int_{\mathbb{R}} s_c(\alpha, \tau) D(\alpha, \tau)\, d\alpha\, d\tau$$

where $s_c \in L^2(\mathbb{R}^2)$.

Suppose that the clutter is characterized as a random field on $\mathbb{R}^2$. That is, the function s_c is considered a random field which is taken to have zero mean and covariance function $R((\alpha, \tau), (\alpha', \tau'))$. As such, s_c has a Karhunen-Loève expansion [1, p. 71] in terms of the eigenfunctions of the covariance operator

$$s_c(\alpha, \tau) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j \phi_j(\alpha, \tau)$$

where $\mathsf{E}(\xi_j) = 0$ and $\mathsf{E}(\xi_j \overline{\xi_k}) = \delta_{jk}$ for $j, k = 1, \cdots, \infty$. Here the ϕ_j are the orthonormal eigenfunctions of the covariance operator; i.e.,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} R((\alpha, \tau), (\alpha', \tau'))\phi_j(\alpha', \tau')\, d\alpha' d\tau' = \lambda_j \phi_j(\alpha, \tau)$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \phi_j(\alpha, \tau)\overline{\phi_k(\alpha, \tau)}\, d\alpha\, d\tau = \delta_{jk}$$

Defining the clutter eigen-mode operators Γ_j by

$$\Gamma_j = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_j(\alpha, \tau) D(\alpha, \tau)\, d\alpha\, d\tau, \tag{3}$$

they satisfy

$$\mathrm{Tr}(\Gamma_j \Gamma_k^\dagger) = \delta_{jk}$$

and the clutter operator can be written as

$$C = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j \Gamma_j \tag{4}$$

In practice, only a certain number of the eigenvalues will be appreciable and the sum can be truncated. Note that this clutter model does not assume any form of stationarity for the clutter.

Stochastic target models can be developed in a similar way in terms of a localized random field. With such a model, S would take the form of a stochastic linear combination of orthonormal target eigen-mode operators Λ_j with associated eigenvalues μ_j

$$S \approx \left[\sum_{j=1}^{M} \sqrt{\mu_j} \xi_j \Lambda_j \right] D(\alpha_0, \tau_0)$$

where M is an appropriate truncation point.

3. DETECTION IN AWGN

Consider a monostatic radar with baseband waveform $w \in L^2(\mathbb{R})$. At the carrier frequency Ω, the waveform is $D(\Omega, 0)w$. Employing a H-S scene model with one illuminated point scatterer at range r and radial velocity v relative to the radar, the return is $s(\alpha_0, \tau_0)D(\alpha_0, \tau_0)[D(\Omega, 0)w]$ where $\tau_0 = 2r/c$ and $\alpha_0 \approx 2v\Omega/c$ (c is the speed of light). The process of mixing the received signal back to baseband yields

$$\begin{aligned} x(t) &= D(-\Omega, 0)s(\alpha_0, \tau_0)D(\alpha_0, \tau_0)D(\Omega, 0)w(t) \\ &= e^{-i\Omega\tau_0} e^{i\alpha_0 t} s(\alpha_0, \tau_0) w(t - \tau_0) \\ &= s(\alpha_0, \tau_0) e^{-i\Omega\tau_0} D(\alpha_0, \tau_0) w(t) \end{aligned}$$

For an extended target in zero-mean additive white Gaussian noise $n(t)$, (2) gives

$$\begin{aligned} x(t) &= \sum_{(\alpha,\tau)} e^{-i\Omega\tau} s_t(\alpha, \tau) D(\alpha, \tau) w(t) + n(t) \\ &= Tw(t) + n(t) \end{aligned} \tag{5}$$

where the H-S class target response operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$T = \sum_{(\alpha,\tau)} e^{-i\Omega\tau} s_t(\alpha, \tau) D(\alpha, \tau)$$

To test the hypothesis H_1 that the received signal x has the form (5) against the null hypothesis H_0 that $x(t) = n(t)$, observe that under H_1, $x(t)$ is Gaussian with mean $Tw(t)$ and variance σ^2 (i.e., the variance of the noise process n). Under H_0, $x(t)$ is Gaussian with mean zero and variance σ^2. Due to the Gaussian nature of this problem, detection performance (e.g., in the Neyman-Pearson sense) [3] is optimized by maximizing the signal-to-noise ratio, which is equivalent to maximizing

$$||Tw||^2 = \langle Tw, Tw \rangle = \langle T^\dagger Tw, w \rangle$$

where $T^\dagger$ denotes the adjoint of T. Hence detection performance is optimized for a given signal energy by choosing the waveform w to be an eigenfunction of the non-negative definite Hermitian operator $P = T^\dagger T$ corresponding to its largest eigenvalue.

If the constraint is power rather than energy (i.e., with $|w(t)| < B$ for all t), it is generally possible to improve on this eigenfunction solution. This analysis will be presented in a later paper.

4. DETECTION IN CLUTTER AND NOISE

As in the preceding section, assume a target has response operator T which is in the H-S class in $L^2(\mathbb{R})$ and that the noise is white and Gaussian with variance σ^2. In addition, the transmitted signal w is affected by a H-S class clutter operator C. The signal presented to the receiver is thus

$$s(t) = Tw(t) + Cw(t) + n(t)$$

The receiver processes s with a H-S operator H and makes its detection decision on the basis of the statistic $Hs(t_0)$, the response of this operator at a particular instant t_0 (called the "decision instant" in [5]). This value is given by $r(t_0) = (HTw + HCw + Hn)(t_0)$ and has associated signal to interference plus noise ratio

$$\mathrm{SINR}(t_0) = \frac{|HTw(t_0)|^2}{\mathsf{E}|HCw(t_0) + Hn(t_0)|^2} \tag{6}$$

As in [5], the initial objective is taken to be maximizing the SINR at time t_0, given w, by suitable choice of H. Ultimately, once the optimal H is known as a function of w, the waveform that maximizes SINR when paired with its corresponding optimal receiver processing H will be sought. This second optimization step is only discussed superficially in this paper, however.

As a H-S operator on $L^2(\mathbb{R})$, H may be represented as an integral operator with kernel $\Phi : \mathbb{R}^2 \to \mathbb{C}$; i.e., for any $f \in L^2(\mathbb{R})$,

$$Hf(t) = \int_{\mathbb{R}} \Phi(t,\tau) f(\tau)\, d\tau$$

Consequently,

$$\begin{aligned}
\mathsf{E}|Hn(t)|^2 &= \mathsf{E}\left[\int \Phi(t,\tau)n(\tau)d\tau \overline{\int \Phi(t,u)n(u)du}\right] \\
&= \int\int \Phi(t,\tau)\overline{\Phi(t,u)}\mathsf{E}[n(u)n(\tau)]du\, d\tau \\
&= \sigma^2 \int |\Phi(t,\tau)|^2\, d\tau
\end{aligned}$$

Denoting $h_{t_0} = \Phi(t_0,\cdot)$, the properties of the transform kernel imply $h_{t_0} \in L^2(\mathbb{R})$, and the preceding expression gives

$$\mathsf{E}|Hn(t_0)|^2 = \sigma^2 ||h_{t_0}||^2 \tag{7}$$

For the clutter term,

$$\begin{aligned}
\mathsf{E}|HCw(t_0)|^2 &= \int\int h_{t_0}(\tau)\overline{h_{t_0}(u)}Cw(\tau)\overline{Cw(u)}\; d\tau\, du \\
&= \langle h_{t_0}, G_C h_{t_0}\rangle \qquad (8)
\end{aligned}$$

where G_C is a non-negative definite Hermitian operator defined in terms of the waveform w and the clutter operator C by

$$G_C f(t) = \int_{\mathbb{R}} \mathsf{E}[\overline{Cw(t)}Cw(\tau)]\, f(\tau)\; d\tau$$

The presence of the expectation operator in this expression indicates that the clutter operator may be regarded as random. If this is the case, the ensuing analysis is unaffected provided that the noise n has zero mean (as already assumed) and is independent of C. For deterministic (known) clutter

$$G_C = CP_wC^\dagger$$

where P_w is the rank-one projection operator from $L^2(\mathbb{R})$ onto the one-dimensional subspace spanned by w; i.e., for $f \in L^2(\mathbb{R})$.

$$P_w f = \langle f, w\rangle\, w$$

In the stochastic case the expansion (4) gives

$$\begin{aligned}
G_C &= \mathsf{E}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sqrt{\lambda_j\lambda_k}\xi_j\xi_k\Gamma_j P_w \Gamma_k^\dagger\right) \\
&= \sum_{j=1}^{\infty}\lambda_j\Gamma_j P_w\Gamma_j^\dagger.
\end{aligned}$$

Finally, the signal term in the numerator of (6) may be written in inner product form as

$$|HTw(t_0)|^2 = \langle h_{t_0}, P_{Tw}h_{t_0}\rangle \tag{9}$$

where P_{Tw} is the rank-one projection operator from $L^2(\mathbb{R})$ onto the one-dimensional subspace spanned by Tw. Substituting (7), (8), and (9) into (6) yields

$$\mathrm{SINR}(t_0) = \frac{\langle h_{t_0}, P_{Tw}h_{t_0}\rangle}{\langle h_{t_0}, (\sigma^2\mathbb{I} + G_C)h_{t_0}\rangle} \tag{10}$$

where $\mathbb{I}$ denotes the identity operator on $L^2(\mathbb{R})$. In this form, maximization of SINR may be treated as a generalized eigenvalue problem. The operator P_C in the numerator is positive definite and of rank one. Since G_C is non-negative definite, the operator $(\sigma^2\mathbb{I}+G_C)$ in the denominator is positive definite as long as $\sigma^2 > 0$. Thus SINR is maximized by setting

$$h_{t_0} = h_{\max} = (\sigma^2\mathbb{I} + G_C)^{-1/2}Tw$$

To calculate the maximal SINR value, denote $G = (\sigma^2\mathbb{I}+ G_C)$ and observe that G is positive definite and self-adjoint. Thus $h_{\max} = G^{-1/2}Tw$ and the maximal value of $\mathrm{SINR}(t_0)$ is

$$\mathrm{SINR}_{\max} = \frac{\left\langle G^{-1/2}Tw, P_{Tw}G^{-1/2}Tw\right\rangle}{\left\langle G^{-1/2}Tw, G^{1/2}Tw\right\rangle}$$

Denoting $f = Tw$ and using the self-adjointness of G gives

$$\begin{aligned}
\mathrm{SINR}_{\max} &= \frac{\left\langle G^{-1/2}f, P_{Tw}G^{-1/2}f\right\rangle}{\left\langle G^{-1/2}f, G^{1/2}f\right\rangle} \\
&= \frac{\left\langle f, G^{-1/2}P_{Tw}G^{-1/2}f\right\rangle}{\langle f, f\rangle}
\end{aligned}$$

The maximal value of this ratio is exactly the maximal eigenvalue of the non-negative definite operator $G^{-1/2}P_{Tw}G^{-1/2}$. Since this is a rank-one operator, its trace is equal to its one positive eigenvalue. Hence,

$$\begin{aligned}
\mathrm{SINR}_{\max} &= \mathrm{Tr}(G^{-1/2}P_{Tw}G^{-1/2}) \\
&= \left\langle G^{-1/2}f, G^{-1/2}f\right\rangle \\
&= \left\langle f, G^{-1}f\right\rangle = \left\langle Tw, G^{-1}Tw\right\rangle \\
&= \left\langle Tw, (\sigma^2\mathbb{I} + G_C)^{-1}Tw\right\rangle
\end{aligned}$$

Note that this agrees with [6, Eq. (11)] when the noise is white and for the special case of LTI operators.

The waveform design problem is reduced to finding

$$w_{\max} = \arg\max_{w\in W}\left\langle w, T^\dagger(\sigma^2\mathbb{I} + G_C)^{-1}Tw\right\rangle \tag{11}$$

where W is a permissible class of transmit waveforms. Note that this is a non-linear problem because the operator G_C depends on w. When the clutter operator C is known (deterministic) then

$$(\sigma^2\mathbb{I} + G_C)^{-1} = \frac{1}{\sigma^2}\left(\mathbb{I} - \frac{CP_wC^\dagger}{\langle w, (C^\dagger C + \sigma^2\mathbb{I})w\rangle}\right)$$

and so the design problem becomes

$$w_{\max} = \arg\max_{w\in W}\left(\left\langle w, T^\dagger Tw\right\rangle - \frac{|\left\langle w, T^\dagger Cw\right\rangle|^2}{\langle w, (C^\dagger C + \sigma^2\mathbb{I})w\rangle}\right)$$

5. CONCLUSIONS AND FUTURE WORK

This paper has proposed modeling of radar scenes by Hilbert-Schmidt class operators on $L^2(\mathbb{R})$, arguing that this class of models extends the commonly used LTI class of operator models to accommodate both time delay and frequency shifts. Some remarks on stochastic clutter modeling within this framework were also presented.

If a target with known operator model is to be detected in additive white Gaussian noise, the optimal waveform subject to a total energy constraint has been observed to arise as an eigenfunction of an operator formed from the target model operator. In the presence of a known clutter operator, the receiver processing that maximizes the SINR at a chosen time instant was shown to be characterized by the solution to a generalized eigenvalue problem.

Future work will investigate optimization of the waveform in the presence of clutter and subject to an energy constraint. Investigation of the use of this operator-theoretic perspective to optimize joint waveform and processing design under power rather than energy constraints, as mentioned in Section 3, is also underway.

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