A HIERARCHICAL APPROACH TO HIGH RESOLUTION RADAR

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ABSTRACT

Radar is usually carried out in a regime where targets are to be resolved and detected in clutter. In this regime the ambiguity properties of the waveform and the uncertainty principle reign supreme. However, for certain applications such as space situation awareness, the problem becomes one of resolving a small (but unknown) number of targets in noise. We present a hierarchical approach to this problem. The approach starts with a one target model for detection and then uses a two (and perhaps more) target model to resolve closely space targets. We consider bound on the resolution of closely spaced target through the computation of the Fisher information metric and compare the results with a number recent papers in compressive radar.

***Index Terms*—** Maximum likelihood, high resolution radar

1. INTRODUCTION

Radar is usually carried out in a regime where targets are to be resolved and detected in clutter. A prior there is very little knowledge of the distribution of clutter and one has only some rough idea of how the target might be distinguished from the clutter. For example, targets of interest might be distinguished from background clutter is either through having a larger cross-section than the surrounding clutter, by having a different statistical behavior from the clutter, or by being separated, perhaps through appropriate choice of transmit waveform, from the clutter in the range-Doppler plane.

In some applications such as space situational awareness (SSA) one might have a substantial of prior information, i.e., "in some region of interest there are a small, but unknown, number of essentially point targets". In such a scenario, for a transmit waveform f, the base-band return from k targets can be modelled as

$$r(t) = \sum_{\ell=1}^{k} a_\ell f(t-\tau_\ell) e^{i\omega_\ell t} + \nu(t) \tag{1}$$

where a_ℓ, τ_ℓ and ω_ℓ are the scattering amplitude, delay and Doppler of the ℓ^{th} target and $n(t)$ is Gaussian white noise. This is just a parameter estimation problem which can be handled conventional statistical techniques, the main difficulty being in estimating the number of targets. For given the number of targets one could find the maximum likelihood estimate of the parameters given the number of targets, although this could be a time consuming process.

We address two problem in this paper. The first is how the resolution of a radar depends of prior infomation about the radar scene. The second is how to get most of the benefits of this general multiple target model based approaches such as full ML, or maximum a posterior distribution (MAP), while avoiding the drawbacks of the necessity of knowing the number of targets, as well as the computational complexity. We develop an approach to the radar target detection/estimation problem based on an idea of Bretthorst [**?**], which he applied to the Bayesian detection and estimation of multiple discrete sinusoids in noise.

In our approach we initially analyse the return using the one target model. That is, we first look for activity in the range-Doppler plane using the Bayesian one target model. This is akin to analysis of a signal comprised of a finite linear combination of sinusoids using the periodogram [**?**]. Range-Doppler regions in which target activity is detected using the one signal posterior distribution can be further analyzed and resolved using a two target model, but the analysis is only carried out locally, that is, within the range-Doppler region covered by a single peak in the one signal posterior distribution. Since we only search locally within the two target posterior distribution the computational burden is kept manageable.

2. PRIOR INFORMATION AND RADAR SIGNAL PROCESSING

Traditionally the radar detection problem starts with a paucity of information about the radar scene to be observed. A finite energy waveform (usually narrowband) $f \in L^2(\mathbb{R}^2)$ is transmitted and the resulting reflection from the environment or radar scene is observed. We can think of the radar scene as on operator S on the space of waveforms $L^2(\mathbb{R}^2)$ and our goal in radar is to determine as much as we can about S from it action on a single element of $L^2(\mathbb{R}^2)$. Furthermore, the receiver waveform is usually contaminated with a significant amount of receiver noise. That is, our problem is to determine S from

$$r = Sf + \nu \tag{2}$$

where ν is narrow band Gaussian noise process. For a single point scatter with delay the transmitted signal by τ and Doppler shifts it by ω, we have $S = aD(\tau, \omega)$ where

$$\left(D(\tau, \omega)f\right)(t) = e^{-i\tau\omega/2} e^{i\omega t} f(t - \tau) \tag{3}$$

for any $f \in L^2(\mathbb{R}^2)$ and a is a complex parameter which includes the effect of the radar cross-section of the point target and the gain and attenuation effect of range and antennas. Assuming that S is Hilbert-Schmidt class, i.e. $\operatorname{Tr} SS^\dagger < \infty$, it can be written as

$$S = \int_{\mathbb{R}^2} a(\tau, \omega) D(\tau, \omega)\, d\tau d\omega \tag{4}$$

where the function $a \in L^2(\mathbb{R}^2)$. In the absence of noise we now know

$$S = SP_f + R, \tag{5}$$

where R satisfies $Rf = 0$, but is otherwise is undetermined and P_f is the orthogonal projection onto the one dimensional subspace spanned by f. Since the transmitted waveform tells us nothing about the operator R, we obtain an estimate of the scene

$$\hat{S} = SP_f = P_{rf}, \tag{6}$$

where

$$P_{rf} = (r, f) r, \tag{7}$$

We can expand $\hat{S}$ in terms of its scattering distribution

$$\hat{S} = \int_{\mathbb{R}^2} \hat{a}(t, \omega) D(t, \omega)\, dt d\omega, \tag{8}$$

where

$$\hat{a}(t, \omega) = (D(\tau, \omega)f, r), \tag{9}$$

which is the output of the matched filter and is related to the true scattering distribution by

$$\hat{a}(\tau, \omega) = \int_{\mathbb{R}^2} a(\tau', \omega') \overline{\mathcal{A}_f(\tau' - \tau, \omega' - \omega)} e^{i(\omega\tau' - \omega'\tau)}\, d\tau' d\omega'. \tag{10}$$

where

$$\mathcal{A}_f(\tau, \omega) = (f, D(\tau, \omega) f) \tag{11}$$

is the *ambiguity* function of the waveform f. In the absence of additional prior information about the form of the radar scene this is the best we can do. Equation (10) implies that the ambiguity function acts a complex point-spread function on the actual radar scene.

The way that targets of interest are distinguished from background clutter is either through having a larger cross-section than the surrounding clutter or by being separated, perhaps through appropriate choice of transmit waveform, from the clutter in the range-Doppler plane. The radar target detection/estimation problem can loosely classified as follows:

1. A single target which is in a region of the range-Doppler plane separated from the influence of other scatters.

2. A small number of targets in a region of the range-Doppler plane separated from the influence of other scatters.

3. Detection/estimation in clutter.

The first scenario is very well understood [cite van Trees]. The is detection of a single target in receiver noise (Gaussian and white). The radar return in this case is

$$r(t) = af(t - \tau) e^{i\omega t} + \nu(t) \tag{12}$$

where the complex amplitude a has a Swerling I target prior, i.e., $a \sim \mathcal{CN}(0, \sigma_a^2)$. In this case the matched filter output

$$\Lambda(\tau, \omega) = (D(\tau, \omega) f, r) = \int_{-\infty}^{\infty} \overline{f(t - \tau)} e^{-i\omega t} r(t)\, dt \tag{13}$$

For optimal detection maximum value of $|\Lambda(\tau, \omega)|^2$ is compared to some threshold and the ML estimate of the delay and Doppler of the target is

$$(\hat{\tau}, \hat{\omega}) = \underset{(\tau, \omega)}{\arg\max}\, |\Lambda(\tau, \omega)|^2. \tag{14}$$

The expected potential accuracy of an estimate is determined by the Fisher information (**??**). For a waveform which gives some degree of localization in range and Doppler, the accuracy of estimate of (τ, ω) ultimately depends on the signal to noise ratio, i.e., by increasing the SNR the expected accuracy of the estimates can be made as small as we like.

In the second scenario the radar return has the form

$$r(t) = \sum_{\ell=1}^{k} a_\ell f(t - \tau_\ell) e^{i\omega_\ell t} + \nu(t) \tag{15}$$

This is the situation one would find in a space situational awareness application. What we have is a parameter estimation problem for the parameters (τ_ℓ, ω_ℓ), $\ell = 1, \cdots, k$. It is this problem that we will at least partially address in the paper. This is a parameter estimation problem and so for a given number of targets k the achievable resolution is determined by the Fisher information and not by the uncertainty principle, although if k is too large or the SNR too low, it is possible that global ambiguities may arise.

Finally the third scenario, the one in which most radars need to work, we are in imaging mode and (10) along with the uncertainty principle apply. At interesting feature of radar is that scenarios 1 and 3 in which we have radically different prior information the appropriate thing to do is look directly at the output of the matched filter $|\Lambda(\tau, \omega)|^2$. In the second scenario we will see that the matched filter still appears but that sufficient statistics are constructed by comparing multiple points in the matched filter output.

Our point in this discussion is that to understanding the constrains put on the resolution of radar targets by the radar uncertainty principle one must clearly understand the nature of the prior information available.

3. RELATION TO COMPRESSIVE SENSING

A recent paper [cite Strohmer] has address the problem of "High-Resolution Radar via Compressed Sensing". There approach is based the following discrete model of radar. The radar scene consists of point scatters which occupy distinct point a the lattice $\mathbb{Z}_p \times \mathbb{Z}_p$ for some large $p \in \mathbb{Z}$. Waveforms are represented as functions on $L^2(\mathbb{Z}_p)$. A unit point scatter at $(\mu, \nu) \in \mathbb{Z}_p \times \mathbb{Z}_p$ is represented by the linear operator $W(\mu, \nu)$ on $L^2(\mathbb{Z}_p)$ with action

$$(W(\mu, \nu)f)_n = \zeta^{\nu n} f_{n-\mu}, \tag{16}$$

where $\zeta = e^{2\pi i/p}$. The operators

$$\mathcal{B} = \{W(\mu, \nu) | (\mu, \nu) \in \mathbb{Z}_p \times \mathbb{Z}_p\} \tag{17}$$

form a basis for the space of linear operators on $L^2(\mathbb{Z}_p)$ as so any radar scene S (in this model) can then be written as

$$S = \sum_{(\mu,\nu)\in\mathbb{Z}_p\times\mathbb{Z}_p} a_{(\mu,\nu)} W(\mu, \nu). \tag{18}$$

The compressive sensing aspect come in when one has prior knowledge that the radar scene is sparse in the basis $\mathcal{B}$, i.e., there are a small number k of targets which lay in $\mathbb{Z}_p \times \mathbb{Z}_p$, so that

$$S = \sum_{\ell=1}^{k} a_\ell W(\mu_\ell, \nu_\ell) \tag{19}$$

where $k \ll p^2$. If we transmit a waveform f the the radar return is

$$\begin{aligned} r_n &= \sum_{\ell=1}^{k} a_\ell \zeta^{\nu_\ell n} f_{n-\mu_\ell} \\ &= \sum_{(\mu,\nu)\in\mathbb{Z}_p\times\mathbb{Z}_p} \Phi_{n,(\mu,\nu)} a_{(\mu,\nu)}. \end{aligned} \tag{20}$$

which should be compared with (15). The second line in this equation cast the problem in the usual compressive sensing form. Φ is a matrix with $(\mu, \nu)^{\text{th}}$ column $W(\mu, \nu)f$ and $a = [a_{(\mu,\nu)}]$ is an $p \times p$ vector with at most k non-zeros entries. The attraction of compressive sensing is that if k is sufficiently small compared to p identifiability in (20) is guaranteed, i.e., the $(a_\ell, \mu_\ell, \nu_\ell)$ can be recovered uniquely from the data r. This can further be extended to the case where noise with a hard limited amplitude is added to (20). The price to be paid for this identifiability is two fold. Firstly, the model used for the radar scene means that little can be inferred about the resolution of real radars and we discuss this point in more detail below. Secondly, as we discussed previously, if our prior knowledge is that there are only a few targets in an isolated region of the range-Doppler plane, then depending on the waveform and SNR, the radar can have potentially unlimited resolution. This point is missed by this discrete model.

A portion of this discrete model can be considered an approximation to an actual radar. In a modern radar we have sampled data at the receiver and processing is often done (e.g. the FFT) within a discrete periodic framework. These are aspects of the radars operation that the radar engineer has control over. The aspect of this radar model which is not at all realistic is that the radar targets not only lie at the points of a grid, but on a grid related to the sampling interval at the receiver. The radar engineer has no control over the positioning of the targets and the positioning of target on a grid is quite artificial.

The effect of point target at any real delay and Doppler can be included in this model as follows. A unit point scatter $(\mu, \nu) \in \mathbb{Z}_p \times \mathbb{Z}_p$ is represented by the linear operator $D(\tau, \omega)$ on $L^2(\mathbb{Z}_p)$ with action

$$(D(\tau, \omega)f)_n = e^{i\omega n}(f * \phi_\tau)_n, \tag{21}$$

where $\phi_\tau \in L^2(\mathbb{Z}_p)$ is given by

$$(\phi_\tau)_n = e^{\pi i(p-1)\tau/p} \frac{\sin(2\pi\tau)}{\sin(2\pi\tau/p)}. \tag{22}$$

and $*$ denotes convolution on $\phi_\tau \in L^2(\mathbb{Z}_p)$. Note that if $\tau = \mu$ and $\omega = 2\pi\nu/p$ with $\mu, \nu \in \mathbb{Z}_p$, then $D(\tau, \omega) = W(\mu, \nu)$.

A k-"sparse" radar scene now takes the form

$$S = \sum_{\ell=1}^{k} a_\ell D(\tau_\ell, \omega_\ell) \tag{23}$$

Here the radar scene is sparse, but not in the compressive sensing definition of being sparse in some basis. It is sparse in the uncountable set

$$\mathcal{D} = \{D(\tau, \omega) \mid \tau \in [0, 1), \omega \in [0, 2\pi)\} \tag{24}$$

The current compressive sensing theory does not address this situation.

4. ANALYSIS MODELS

Our goal in this paper is to estimate the develop computationally efficient method for estimating the range and Doppler of an unknown but small number of targets k and to understand the potential accuracy of such estimates

As we have said our approach we initially analyse the return using the one target model. That is, we first look for activity in the range-Doppler plane using the Bayesian one target model. Range-Doppler regions in which target activity is detected using the one signal posterior distribution can be further analyzed and resolved using a two target model, but the analysis is only carried out locally, that is, within the range-Doppler region covered by a single peak in the one signal posterior distribution. In this section we develop and analyse the one and two-target models.

4.1. One Target Model

In this section we recap the detection of a single point target in Gaussian white noise [cite Van Trees]. We include it here to set the stage for the discussion of the two-target model and because our approach is a little different.

The transmitted signal form can be expressed as

$$s(t) = f(t)e^{i\omega_c t}, \tag{25}$$

where $f(t)$ is the complex modulation function, ω_c is carrier frequency. The demodulated reflected signal can be represented as

$$r(t) = af(t-\tau)e^{i\omega t} + n(t), \tag{26}$$

where τ is time delay and ω is the Doppler shift, $a \in \mathbb{C}$, and the additive zero mean complex Gaussian white noise process

$$E(n(t)\overline{n(t')}) = \sigma^2\delta(t-t'). \tag{27}$$

Write $f_{\tau,\omega}(t) = f(t-\tau)e^{i\omega t}$. Over the range of τ and ω in the radar scene the set of functions $\{f_{\tau,\omega}\}$, which we take to time-limited and band-limited to within some negligible error, the span$\{f_{\tau,\omega}\}$ is a finite dimensional subspace $V \in L^2(\mathbb{R})$ with dimension N, say. Write the projection of r onto this space as $\tilde{r} = P_V r$ the likelihood function of the parameters (a, τ, ω) given data $\tilde{r}$ is given by

$$p(\tilde{r} \mid a, \tau, \omega) = \frac{1}{(\pi\sigma^2)^N}\exp\left(-\frac{1}{\sigma^2}\|\tilde{r} - af_{\tau,\omega}\|^2\right), \tag{28}$$

Taking a to have a Swerling I prior, i.e., distribution with σ_a^2 with zero mean and variance σ_a^2, ie

$$p(a) = \frac{1}{\pi\sigma_a^2}\exp\left(-|a|^2/\sigma_a^2\right). \tag{29}$$

The marginalised likelihood function of the parameters (τ,ω) is

$$\begin{aligned} p(\tilde{r}|\tau,\omega) &= \int_{\mathbb{C}} p(\tilde{r} \mid a,\tau,\omega)p(a)\,da \\ &= \frac{\sigma^2}{\sigma_a^2\lambda}\frac{1}{(\pi\sigma^2)^N}\exp -\frac{1}{\sigma^2}\left((\tilde{r},\tilde{r}) - \frac{1}{\lambda}|(f_{\tau,\omega},\tilde{r})|^2\right) \end{aligned} \tag{30}$$

where $\lambda = E_f + \sigma^2/\sigma_a^2$, with $E_s = \|f_{\tau,\omega}\|^2$. Note that since $f_{\tau,\omega} \in V$ we have $(f_{\tau,\omega},\tilde{r}) = (f_{\tau,\omega}, r)$. Writing the projector onto $f_{\tau,\omega}$ as $P_{\tau,\omega}$ and the projector onto $\tilde{r}$ as ρ, we can write (30) as

$$p(\tilde{r}|\tau,\omega) = \frac{\sigma^2}{\sigma_a^2\lambda}\frac{1}{(\pi\sigma^2)^N}\exp -\frac{1}{\sigma^2}\mathrm{Tr}\left(\rho(I_V - \frac{1}{\lambda}P_{\tau,\omega})\right) \tag{31}$$

For a uniform prior on (τ,ω) the posterior distribution for (τ,ω) is

$$p(\tilde{r} \mid \tau,\omega) \propto \exp -\frac{1}{\lambda\sigma^2}\mathrm{Tr}\left(\rho P_{\tau,\omega}\right) \tag{32}$$

The likelihood ratio statistic for the hypothesis "there is one target at (τ,ω)" against "there is no target" is the usual

$$\begin{aligned} \frac{p(\tilde{r}|\tau,\omega)}{p(\tilde{r}|\text{no target})} &= \frac{\sigma^2}{\sigma_a^2\lambda}\exp\frac{1}{\lambda\sigma^2}|(f_{\tau,\omega},r)|^2 \\ &= \frac{\sigma^2}{\sigma_a^2\lambda}\exp\frac{1}{\lambda\sigma^2}\mathrm{Tr}(\rho P_{\tau,\omega}) \end{aligned} \tag{33}$$

For the one-target case, writing $(\theta_1,\theta_2) = (\tau,\omega)$, the Fisher information is given by

$$\begin{aligned} \mathcal{I}_{ij}(\tau,\omega) &= -E\left(\frac{\partial^2 L(\tau,\omega)}{\partial\theta_i\partial\theta_j}\right) \\ &= -\frac{1}{\lambda\sigma^2}\mathrm{Tr}\left(\frac{\partial^2 P_{\tau,\omega}}{\partial\theta_i\partial\theta_j}E(\rho)\right) \\ &= -\frac{\sigma_a^2}{\lambda\sigma^2}\mathrm{Tr}\left(\frac{\partial P_{\tau,\omega}}{\partial\theta_i}\frac{\partial P_{\tau,\omega}}{\partial\theta_j}\right) \end{aligned} \tag{34}$$

where $L(\tau,\omega)$ is the log of the likelihood function (30) and we have used

$$E(\rho) = \sigma^2 I_V + \sigma_a^2 P_{\tau,\omega}. \tag{35}$$

Noting that

$$\begin{aligned} P_{\tau,\omega} &= D(\tau,\omega)P_{0,0}D(\tau,\omega)^\dagger \\ &= \iint A_f(\tau',\omega')e^{i(\omega'\tau-\omega\tau')}D(\tau',\omega')\,d\tau' d\omega', \end{aligned} \tag{36}$$

where A_f is the ambiguity function of f and

$$\mathrm{Tr}(D(\tau,\omega)D(\tau',\omega')^\dagger) = \delta(\tau-\tau')\delta(\omega-\omega'), \tag{37}$$

we obtain the result of Van Trees [**?**] in terms of the moments of the Ambiguity function,

$$\mathcal{I}(\tau,\omega) = \frac{\sigma_a^2}{\lambda\sigma^2}\iint |A_f(\tau,\omega)|^2\begin{pmatrix}\omega^2 & \omega\tau \\ \tau\omega & \tau^2\end{pmatrix}d\tau\,d\omega \tag{38}$$

4.2. Two Target Model

Assuming that there are two targets present in noise. The received signal can be written as

$$r(t) = a_1 f(t-\tau_1)e^{i\omega_1 t} + a_2 f(t-\tau_2)e^{i\omega_2 t} + n(t) \tag{39}$$

where $a_1, a_2 \in \mathbb{C}$ are the amplitudes of the reflected signal from the two targets, and $\Theta = \{\tau_1,\tau_2,\omega_1,\omega_2\}$ are the their delay and Doppler shifts. Let

$$f_j(t) = f(t-\tau_j)e^{i\omega_j t}, \quad \text{j = 1, 2.} \tag{40}$$

The likelihood function of the parameters a_1, a_2 and Θ is

$$p(\tilde{r} \mid a_1, a_2, \Theta) = \frac{1}{(\pi\sigma^2)^N} \exp\left(-\frac{1}{\sigma^2}\|\tilde{r} - a_1 f_{\tau_1,\omega_1} - a_2 f_{\tau_2,\omega_2}\|^2\right), \quad (41)$$

Choosing the prior distribution for A to be normal distribution with zero means and variance Σ, that is

$$p(a_1, a_2) = \frac{1}{\pi^2\sigma_1^2\sigma_2^2} \exp\left(-\frac{|a_1|^2}{\sigma_1^2} - \frac{|a_2|^2}{\sigma_2^2}\right), \quad (42)$$

the marginalised likelihood function for the parameters Θ is

$$\begin{aligned} p(\boldsymbol{r}|\Theta) &= \int p(\boldsymbol{r}, |a_1, a_2, \Theta) p(a_1, a_2)\, da_1 da_2 \\ &= \frac{\sigma^4}{(\pi\sigma^2)^N \sigma_1^2\sigma_2^2 |G_\Theta + \sigma^2\Sigma|} \\ &\times \exp -\frac{1}{\sigma^2}\left((\tilde{r}, \tilde{r}) - \boldsymbol{\gamma}_\Theta^\dagger (G_\Theta + \sigma^2\Sigma)^{-1}\boldsymbol{\gamma}_\Theta\right) \end{aligned} \quad (43)$$

where

$$\boldsymbol{\gamma}_\Theta = ((r, f_1), (r, f_2)) \quad (44)$$

is a vector of matched filter outputs,

$$G_\Theta = \begin{pmatrix} E_f & (f_1, f_2) \\ (f_2, f_1) & E_f \end{pmatrix} \quad (45)$$

is the Gramm matrix for the shifted waveforms f_1 and f_2 and

$$\Sigma = \begin{pmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{pmatrix}. \quad (46)$$

The likelihood ratio statistic for the hypothesis ”there are two targets at (τ_1, ω_1) and (τ_2, ω_2)” against ”there is no target” is the usual

$$\frac{p(\tilde{r}|\Theta)}{p(\tilde{r}|\text{no target})} = \frac{\sigma^4}{\sigma_1^2\sigma_2^2|G_\Theta + \sigma^2\Sigma|} \exp\frac{1}{\sigma^2}\left(\boldsymbol{\gamma}_\Theta^\dagger (G_\Theta + \sigma^2\Sigma)^{-1}\boldsymbol{\gamma}_\Theta\right) \quad (47)$$

Comparing this with we see that the matched filter statistic $|(f_{\tau,\omega}, r)|^2$ in the one target detector is replaced by the combination of two matched filter outputs

$$\boldsymbol{\gamma}_\Theta^\dagger (G_\Theta + \sigma^2\Sigma)^{-1}\boldsymbol{\gamma}_\Theta - \sigma^2 \log|G_\Theta + \sigma^2\Sigma| \quad (48)$$

Finally, we note that the expected value of the projector on to $\tilde{r}$ is

$$E(\rho) = \sigma^2 I_V + \sigma_1^2 P_{\tau_1,\omega_1} + \sigma_2^2 P_{\tau_2,\omega_2}. \quad (49)$$

In what follows we will use the abbreviation $P_j = P_{\tau_j,\omega_j}$, for $j = 1, 2$.

4.3. Fisher Information Matrix for Two Target Estimation

Foe the two target case the elements of Fisher information matrix is given

$$\mathcal{I}_{ij} = -E\left[\frac{\partial^2 L(\Theta)}{\partial\theta_i\partial\theta_j}\right], \quad (50)$$

where $\theta_1 = \tau_1$, $\theta_2 = \tau_2$, $\theta_3 = \omega_1$ and $\theta_4 = \omega_2$ and

$$\begin{aligned} L(\Theta) &= -\frac{1}{\sigma^2}\boldsymbol{\gamma}_\Theta^\dagger (G_\Theta + \sigma^2\Sigma)^{-1}\boldsymbol{\gamma}_\Theta - \log|G_\Theta + \sigma^2\Sigma| + \text{const.} \\ &= -\frac{1}{\sigma^2|G_\Theta + \sigma^2\Sigma|}\text{Tr}(\rho Q) - \log|G_\Theta + \sigma^2\Sigma| + \text{const.} \end{aligned} \quad (51)$$

where

$$Q = (E_f + \frac{\sigma^2}{\sigma_2^2})P_1 - P_1P_2 - P_2P_1 + (E_f + \frac{\sigma^2}{\sigma_1^2})P_2 \quad (52)$$

We have for $i = 1, 3$ and $j = 1, 3$

$$\begin{aligned} E\left[\frac{\partial^2 L(\Theta)}{\partial\theta_i\partial\theta_j}\right] &= \frac{\partial^2}{\partial\theta_i\partial\theta_j}|G_\Theta + \sigma^2\Sigma|^{-1}(a_1 + a_2 - b\text{Tr}(P_1P_2)) \\ &- b\frac{\partial}{\partial\theta_i}|G_\Theta + \sigma^2\Sigma|^{-1}\frac{\partial}{\partial\theta_j}\text{Tr}(P_1P_2) \\ &- (1+b)\frac{\partial}{\partial\theta_j}|G_\Theta + \sigma^2\Sigma|^{-1}\frac{\partial}{\partial\theta_i}\text{Tr}(P_1P_2) \\ &+ |G_\Theta + \sigma^2\Sigma|^{-1}\Big(- b\frac{\partial^2}{\partial\theta_i\partial\theta_j}\text{Tr}(P_1P_2) \\ &- \frac{\sigma_1^2}{\sigma^2}\text{Tr}(\frac{\partial P_1}{\partial\theta_i}\frac{\partial P_1}{\partial\theta_j}) + 2\frac{\sigma_1^2}{\sigma^2}\text{Tr}(\frac{\partial P_1}{\partial\theta_i}\frac{\partial P_1}{\partial\theta_j}P_2)\Big) \end{aligned} \quad (53)$$

For $i = 2, 4$ and $j = 2, 4$

$$\begin{aligned} E\left[\frac{\partial^2 L(\Theta)}{\partial\theta_i\partial\theta_j}\right] &= \frac{\partial^2}{\partial\theta_i\partial\theta_j}|G_\Theta + \sigma^2\Sigma|^{-1}(a_1 + a_2 - b\text{Tr}(P_1P_2)) \\ &- b\frac{\partial}{\partial\theta_i}|G_\Theta + \sigma^2\Sigma|^{-1}\frac{\partial}{\partial\theta_j}\text{Tr}(P_1P_2) \\ &- (1+b)\frac{\partial}{\partial\theta_j}|G_\Theta + \sigma^2\Sigma|^{-1}\frac{\partial}{\partial\theta_i}\text{Tr}(P_1P_2) \\ &+ |G_\Theta + \sigma^2\Sigma|^{-1}\Big(- b\frac{\partial^2}{\partial\theta_i\partial\theta_j}\text{Tr}(P_1P_2) \\ &- \frac{\sigma_1^2}{\sigma^2}\text{Tr}(\frac{\partial P_2}{\partial\theta_i}\frac{\partial P_2}{\partial\theta_j}) + 2\frac{\sigma_1^2}{\sigma^2}\text{Tr}(\frac{\partial P_2}{\partial\theta_i}\frac{\partial P_2}{\partial\theta_j}P_1)\Big) \end{aligned} \quad (54)$$

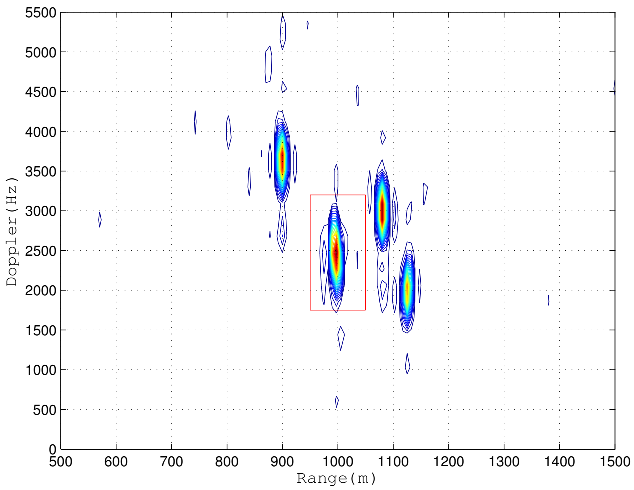


Fig. 1. Output of the one target detector. There are 5 targets at -5 dB SNR. Four peaks are evident.

For $i = 1, 3$ and $j = 2, 4\}$

$$
\begin{aligned}
E\left[\frac{\partial^2 L(\Theta)}{\partial\theta_i\partial\theta_j}\right] &= \frac{\partial^2}{\partial\theta_i\partial\theta_j}|G_\Theta + \sigma^2\Sigma|^{-1}(a_1 + a_2 - b\mathrm{Tr}(P_1P_2)) \\
&- b\frac{\partial}{\partial\theta_i}|G_\Theta + \sigma^2\Sigma|^{-1}\frac{\partial}{\partial\theta_j}\mathrm{Tr}(P_1P_2) \\
&- (1+b)\frac{\partial}{\partial\theta_j}|G_\Theta + \sigma^2\Sigma|^{-1}\frac{\partial}{\partial\theta_i}\mathrm{Tr}(P_1P_2) \\
&+ |G_\Theta + \sigma^2\Sigma|^{-1}\Bigg(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\mathrm{Tr}(P_1P_2) \\
&- 2\mathrm{Tr}(\frac{\partial P_1}{\partial\theta_i}\frac{\partial P_2}{\partial\theta_j}) - 2\frac{\sigma_1^2}{\sigma^2}\mathrm{Tr}(\frac{\partial P_1}{\partial\theta_i}\frac{\partial P_2}{\partial\theta_j}P_1) \\
&- 2\frac{\sigma_2^2}{\sigma^2}\mathrm{Tr}(\frac{\partial P_1}{\partial\theta_i}\frac{\partial P_2}{\partial\theta_j}P_2)\Bigg)
\end{aligned} \tag{55}
$$

where

$$
\begin{aligned}
a_1 &= E_f(1 + \frac{\sigma_1^2}{\sigma^2}E_f)(E_f + \frac{\sigma^2}{\sigma_2^2}) \\
a_2 &= E_f(1 + \frac{\sigma_2^2}{\sigma^2}E_f)(E_f + \frac{\sigma^2}{\sigma_1^2}) \\
b &= (\sigma_1^2 + \sigma_2^2)\frac{E_f}{\sigma^2}
\end{aligned}
$$

5. SIMULATION

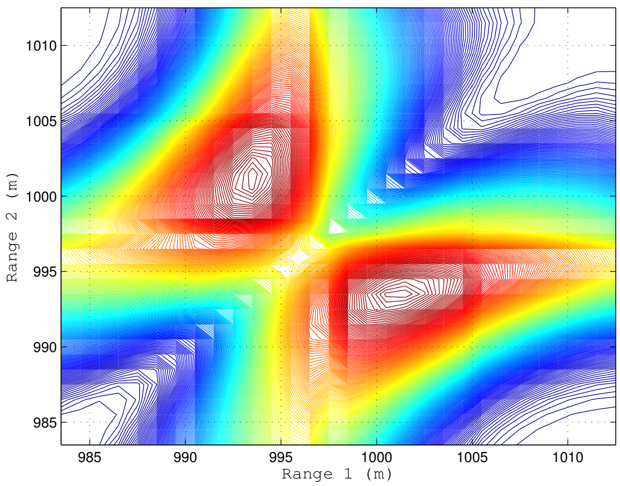
To illustrate our approach we simulated data containing five equal amplitude targets, two of which are closely spaced in range (1000 and 995 meters in range) and Doppler (35 and 38 m/sec) and three well separated both in range and Doppler. The signal-to-noise ratio is -5dB. The simulated pulse train consists of 32 LFM pulses, with pulse width is $5\mu s$ and bandwidth 10 MHz. Given this transmitted waveform, the Range resolution is expected to be 15 meters and the Doppler resolution is expected to be 643 Hz or 9.6m/sec. Figure 1 shows the result of one signal model detector which resolves the four well separated targets. Two of the targets are not resolved. We examine the region surrounding the peak containing the two unresolved targets (the red square in Figure 1) with the two target model detector given in (43). Since the two target model has four parameters, we illustrate the resolving power of the model pictorially by considering the slice $\omega_1 = \omega_2 =$ "the maximum Doppler in the red square". The result is shown in Figure 2. Even though we have the wrong Doppler value a peak corresponding to two targets is clearly observed. Fixing the range values from the maximum in this slice, Figure 3 shows the corresponding Doppler-Doppler plot.

If we apply an optimisation algorithm to the output two target detector, which initialised at the top of the peak in the one target detector we obtain the following estimates for the parameters of the two targets $(1000.70\,\mathrm{m}, 34.1439\,\mathrm{m/sec})$ and $(994.23\,\mathrm{m}, 38.4578\,\mathrm{m/sec})$.

So what happens if we apply this procedure to one of the peaks in Figure 1 which corresponds to a single target? Figure 4 and 5 show the results of applying the above procedure to the single target at range 900 meter and velocity of 55 m/sec. We see that there is no evidence of multiple targets in either plot. Maximisation of the two target detector gives $(900.0000\,\mathrm{m}, 54.7828\,\mathrm{m/sec})$ and $(900.0003\,\mathrm{m}, 54.7723\,\mathrm{m/sec})$ leading to the conclusion that there is one target with paramaters $(900.00\,\mathrm{m}, 54.78\,\mathrm{m/sec})$.

Here we have just given a illustration of the potential resolving power of an hierarchical approach in the situations where our prior knowledge of the radar scene is that there are a small number of targets well separated from clutter. Obviously, one needs to test the performance of this approach over multiple trials and multiple scenarios, these results will be given elsewhere.

6. CONCLUSION



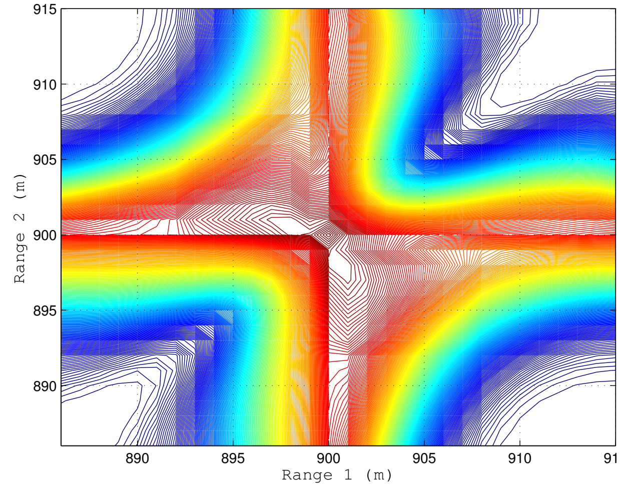
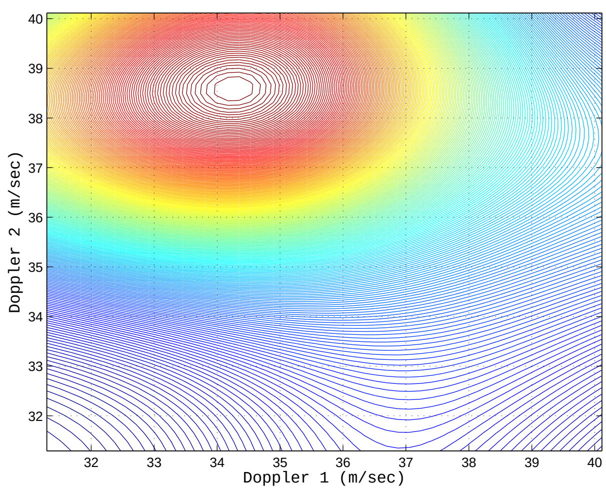


Fig. 2. Plot of the ranges of the two targets around the marked peak in Figure 1 using two target model detector.

Fig. 4. plot of the ranges of single target at range 900m in Figure 1 using two target model detector.



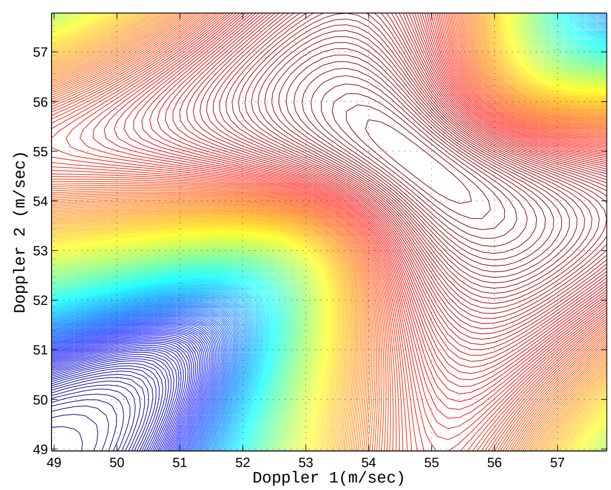


Fig. 3. Plot of the Doppler of the two targets around the marked peak in Figure 1 using two target model detector.

Fig. 5. Plot of the Doppler of single target at velocity 55m/sec in Figure 1 using two target model detector.