Sampling of Homogeneous Polynomials

Somantika Datta (1), Stephen D. Howard (2), and Douglas Cochran (1)

(1) Arizona State University, Tempe, Arizona 85287, USA.
(2) Defence Science & Technology Organisation, Edinburgh, South Australia.
somantika.datta@asu.edu, stephen.howard@dsto.defence.au, cochran@asu.edu

Abstract:

Conditions for reconstruction of multivariate homogeneous polynomials from sets of sample values are introduced, together with a frame-based method for explicitly obtaining the polynomial coefficients from the sample data.

1. Introduction

Several authors have noted the importance of interpolation and reconstruction of multivariate polynomials from sample data in applications. Zakhor [10], for example, considered the problem of interpolation of bivariate polynomials from irregularly spaced sample values in connection with two-dimensional filter design and image processing. The case of multivariate polynomials presents significant difficulties not encountered with polynomials of one variable, in particular due to the zeros of these entire functions of several variables not being isolated as occurs in the univariate setting. Consequently, it is not surprising that, in her work, Zakhor develops conditions in which suitable sampling sets are constrained to lie on certain algebraic curves.

Very recent work by Varjú [9] and Benko and Króo [1] develops Weierstraß types of results for approximation of smooth multivariate functions by homogeneous polynomials. This suggests the potential utility of interpolation and reconstruction of homogeneous polynomials from sample values. It is well known that the linear space $H_k(\mathbb{C}^n)$ of homogeneous polynomials of degree k in n complex variables is isomorphic to the space $\mathrm{Sym}^k(\mathbb{C}^n)$ of symmetric k-tensors over $\mathbb{C}^n$. This fact was used by the authors in [3] to develop results concerning frames and grammians on $\mathrm{Sym}^k(\mathbb{C}^n)$. In this paper, a similar perspective is used to derive conditions under which coefficients of a multivariate homogeneous polynomial of known degree can be reconstructed explicitly from sets of sample values. It is shown that a sampling set that suffices for n-variate homogenous polynomials of degree k is also suitable for reconstructing the coefficients of any homogeneous polynomial in n variables of degree $1 \leqslant \ell < k$. Further, it is noted that, modulo general position issues, the number of samples is the crucial issue in determining suitability of a sampling set. Nevertheless, some sampling sets are "better" than others in that they provide snugger frames and hence the numerical advantages they entail. The relative merits of sampling sets in this respect do not depend on the particular polynomial to be reconstructed, thus allowing generically good sampling sets to be designed before any sampling is actually carried out.

Before beginning the mathematical sections of the paper, a few comments on notation and terminology are in order. For $x = [x^{(1)} \ \cdots \ x^{(n)}]^\mathsf{T}$ and $y = [y^{(1)} \ \cdots \ y^{(n)}]^\mathsf{T}$ in $\mathbb{C}^n$, their inner product will be denoted by

$$\langle x, y\rangle = \sum_{j=1}^{n} \bar{x}^{(j)} y^{(j)}$$

where the bar denotes complex conjugate; i.e., the inner product is conjugate linear in its first argument and linear in its second argument. The corresponding convention will be used for inner products in other complex Hilbert spaces. Given a finite frame $X = \{x_1, ..., x_m\}$ for an n-dimensional complex vector space V, the function $F : V \to \ell_2(\{1, \ldots, m\}) = \mathbb{C}^m$ given by $F(w) = [\langle x_1, w\rangle \ \ldots \ \langle x_m, w\rangle]^\mathsf{T}$ will be called the *frame operator* associated with X, while $\mathcal{F} = F^*F : V \to V$ (i.e., the composition of the adjoint of F with F) will be called the *metric operator* associated with X.

The k-fold tensor product $V^{\otimes k}$ of an n-dimensional vector space V is a vector space spanned by elements of the form $v_1 \otimes \cdots \otimes v_k$ where each $v_i \in V$ [8]. The vector $v_1 \otimes \cdots \otimes v_k$ has n^k coordinates $\{v_i^{(\ell)} | i = 1, \ldots, k;\ \ell = 1, \ldots, n\}$ where $v_i^{(\ell)}$ denotes the ℓ^{th} coordinate of the vector v_i. The space of symmetric k-tensors associated with V, denoted $\mathrm{Sym}^k(V)$, is the subspace of $V^{\otimes k}$ consisting of those tensors which remain fixed under permutation (see Chapter 10 of [8]). $\mathrm{Sym}^k(V)$ is spanned by the tensor powers $v^{\otimes k}$ where $v \in V$. If V has dimension n then $\dim\, \mathrm{Sym}^k(V) = \binom{n+k-1}{k}$. $\mathrm{Sym}^k(V)$ has a natural inner product with the property

$$\left\langle v^{\otimes k}, w^{\otimes k}\right\rangle_{\mathrm{Sym}^k(V)} = \langle v, w\rangle_V^k . \quad (1)$$

2. Sampling of Homogeneous Polynomials

It is well known (see, e.g., [8]) that $H_k(\mathbb{C}^n)$, the linear space of homogeneous polynomials of total degree k in variables $\bar{z}^{(1)}, \ldots, \bar{z}^{(n)}$ is isomorphic to $\mathrm{Sym}^k(V)$. This section points out a connection between the condition that $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is a frame for $\mathrm{Sym}^k(V)$ and the reconstructability of polynomials in $H_k(\mathbb{C}^n)$ from the values they take at sets of m points in $\mathbb{C}^n$.

Beginning with $k = 1$, let $w \in V = \mathrm{Sym}^1(V)$ and denote by $[w^{(1)} \cdots w^{(n)}]^\mathsf{T} \in \mathbb{C}^n$ the coordinates of w in some orthonormal basis for V. There is an obvious isomorphism that takes $w \in V$ to the polynomial $p_w \in H_1(\mathbb{C}^n)$ defined by $p_w(z^{(1)}, \ldots, z^{(n)}) = w^{(1)}\bar{z}^{(1)} + \cdots w^{(n)}\bar{z}^{(n)}$. If $X = \{x_1, \ldots, x_m\}$ is a frame for V, the associated frame operator $F : V \to \mathbb{C}^m$ is given by

$$F(w) = \begin{bmatrix} \langle x_1, w\rangle \\ \vdots \\ \langle x_m, w\rangle \end{bmatrix} = \begin{bmatrix} p_w(x_1^{(1)}, \ldots, x_1^{(n)}) \\ \vdots \\ p_w(x_m^{(1)}, \ldots, x_m^{(n)}) \end{bmatrix}. \quad (2)$$

In other words, $F(w)$ is a vector of values obtained by evaluating (i.e., "sampling") p_w at the points $x_1, \ldots, x_m$. One may ask whether this set of m sample values is sufficient to uniquely determine p_w.

To address this question, define a sampling function $P_X : H_1 \to \mathbb{C}^m$ by

$$P_X(p) = \begin{bmatrix} p(x_1^{(1)}, \ldots, x_1^{(n)}) \\ \vdots \\ p(x_m^{(1)}, \ldots, x_m^{(n)}) \end{bmatrix}$$

and note that (2) shows the frame operator is given by $F(w) = P_X(p_w)$. Because the frame operator is invertible, w is uniquely determined by $F(w)$. Hence any $p_w \in H_1$ is uniquely determined by its samples $P_X(p_w)$. Conversely, if X fails to frame V, the mapping F defined by (2) is still well-defined, but has non-trivial kernel K. In this case, $P_X(p_w) = P_X(p_{w+u})$ for all $u \in K$. So, in particular, p_w is not uniquely determined from its samples at $x_1, ..., x_m$.

A similar situation occurs for $k > 1$, where the space of interest is $\mathrm{Sym}^k(V)$ and the frame is $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$. As in the $k = 1$ case, mapping a polynomial to its coefficient sequence defines an isomorphism between $H_k(\mathbb{C}^n)$ and $\mathrm{Sym}^k(V)$ for $k > 1$. If $v = w^{\otimes k} \in \mathrm{Sym}^k(V)$ is a pure tensor power of $w \in V$, then

$$\begin{aligned} F^{(k)}(v) &= \begin{bmatrix} \langle x_1^{\otimes k}, w^{\otimes k}\rangle \\ \vdots \\ \langle x_m^{\otimes k}, w^{\otimes k}\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle x_1, w\rangle^k \\ \vdots \\ \langle x_m, w\rangle^k \end{bmatrix} = \begin{bmatrix} p_v(x_1) \\ \vdots \\ p_v(x_m) \end{bmatrix} \end{aligned}$$

where $p_v \in H_k$ is defined by $p_v(z) = \langle z, w\rangle^k$. $\mathrm{Sym}^k(V)$ is spanned by pure tensor powers of elements in V [8]. Thus, for arbitrary $v \in \mathrm{Sym}^k(V)$, $F^{(k)}(v)$ is a vector of m samples of a polynomial in H_k taken at points $x_1, ..., x_m$. Thus, as in the $k = 1$ case, polynomials in H_k are uniquely determined by the samples $P_X^{(k)}(p) = [p(x_1), \ldots, p(x_m)]^\mathsf{T}$ if and only if $X^{(k)}$ frames $\mathrm{Sym}^k(V)$. Theorem 1 given below implies that if one can reconstruct a polynomial in $H_k(\mathbb{C}^n)$ from a certain sampling set then the same set can be used to reconstruct polynomials in $H_\ell(\mathbb{C}^n)$ for all $1 \leqslant \ell < k$. Conversely, almost every sampling set in $\mathbb{C}^n$ for H_1 gives rise to a sampling set for H_k where $k > 1$, provided there are enough vectors in the set.

Theorem 1. *(i) Given n and m with $m \geqslant n$, if $X^{(k)} = \{x_1^{\otimes k}, x_2^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is a frame for $\mathrm{Sym}^k(V)$, then $X^{(\ell)}$ is a frame for $\mathrm{Sym}^\ell(V)$ for all $1 \leqslant \ell < k$.*
(ii) Almost every set of m vectors in $\mathbb{C}^n$ such that $m \geqslant \binom{n+k-1}{k}$ results in a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$ for $k > 1$.

Proof. (i) Suppose that $X^{(\ell)}$ is not a frame for $\mathrm{Sym}^\ell(V)$. Then $X^{(\ell)}$ does not span $\mathrm{Sym}^\ell(V)$ and there exists $g \in (\mathrm{span}(X^{(\ell)}))^\perp \subset \mathrm{Sym}^\ell(V)$. Take some $h \in \mathrm{Sym}^{k-\ell}(V)$. Let $h = x_1^{\otimes(k-\ell)}$. Then

$$\begin{aligned} \left\langle g \otimes h, x_i^{\otimes k}\right\rangle &= \left\langle g \otimes h, x_i^{\otimes \ell} \otimes x_i^{\otimes(k-\ell)}\right\rangle \\ &= \left\langle g, x_i^{\otimes \ell}\right\rangle \left\langle h, x_i^{\otimes(k-\ell)}\right\rangle \\ &= 0 \cdot \left\langle x_1^{\otimes(k-\ell)}, x_i^{\otimes(k-\ell)}\right\rangle = 0 \end{aligned}$$

which is a contradiction since $g \otimes h \in \mathrm{Sym}^k(V)$ and $X^{(k)}$ is a frame for $\mathrm{Sym}^k(V)$ so that for any i, $\left\langle g \otimes h, x_i^{\otimes k}\right\rangle$ cannot be zero.

(ii) It has been shown in [7] that for almost every set of vectors $X = \{x_1, \ldots, x_m\}$ in $\mathbb{C}^n$, the rank of the grammian of $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is $\binom{n+k-1}{k}$ when $m > \binom{n+k-1}{k}$. This means that the maximum number of linearly independent vectors in $X^{(k)}$ is $\binom{n+k-1}{k}$ which is the same as the dimension of $\mathrm{Sym}^k(\mathbb{C}^n)$ and hence $X^{(k)}$ is a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. □

3. Illustrative Examples

Example 1. Consider the space $V = \mathbb{C}^2$ over the field $\mathbb{C}$. Let $x_1 = [1, 0]^\mathsf{T}$, $x_2 = [0, 1]^\mathsf{T}$ and $x_3 = [1, 1]^\mathsf{T}$. The set $X = \{x_1, x_2, x_3\}$ is a frame for V with corresponding frame operator

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The metric operator is

$$\mathcal{F} = F^*F = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues of $\mathcal{F}$ are 1 and 3, which are the optimal lower and upper frame bounds respectively.

$$\mathcal{F}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

which is the metric operator of the dual frame The dual frame is denoted by $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$ where

$$\tilde{x}_1 = \mathcal{F}^{-1}x_1 = \left[\frac{2}{3}, -\frac{1}{3}\right]^\mathsf{T},$$

$$\tilde{x}_2 = \mathcal{F}^{-1}x_2 = \left[-\frac{1}{3}, \frac{2}{3}\right]^\mathsf{T}, \text{ and}$$

$$\tilde{x}_3 = \mathcal{F}^{-1}x_3 = \left[\frac{1}{3}, \frac{1}{3}\right]^\mathsf{T}.$$

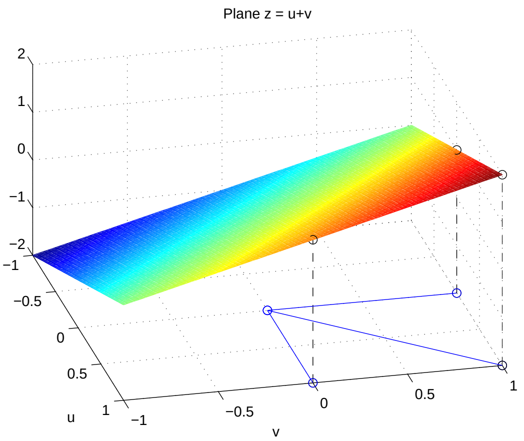

Figure 1: The plane $z = u+v$, a homogeneous polynomial of degree one.

Consider reconstruction of the homogeneous polynomial p of degree one in two variables defined by $p(u,v) = c^{(1)}u + c^{(2)}v$ from the three frame elements. Here $k = 1$, $n = 2$ and $m = 3$. Any $c = [c^{(1)}, c^{(2)}]^\mathsf{T} \in \mathbb{C}^2$ can be reconstructed via the frame reconstruction formula

$$c = \sum_{i=1}^{3} \langle x_i, c\rangle \, \tilde{x}_i. \tag{3}$$

Since $p(x_i) = \langle x_i, c\rangle$ and for this example $p(x_1) = c^{(1)}$, $p(x_2) = c^{(2)}$, and $p(x_3) = c^{(1)} + c^{(2)}$, the right side of (3) is

$$c^{(1)}\tilde{x}_1 + c^{(2)}\tilde{x}_2 + (c^{(1)} + c^{(2)})\tilde{x}_3 = [c^{(1)}, c^{(2)}]^\mathsf{T}.$$

This shows that the coefficients of $p(u,v)$ can be reconstructed from its samples at the frame elements. The polynomial $p(u,v) = u + v$ together with the sampling set is shown in Figure 1.

Example 2. If the homogeneous polynomial to be reconstructed is of degree two as given by $p(u,v) = c^{(1)}u^2 + c^{(2)}uv + c^{(3)}v^2$ then one considers the space $\mathrm{Sym}^2(\mathbb{C}^2) \subset \mathbb{C}^{\otimes 2}$. The dimension of $\mathrm{Sym}^2(\mathbb{C}^2)$ is three, which is the same as the dimension of $H_2(\mathbb{C}^2)$. Hence at least three sampling points are needed. Consider the same set of sampling points as in Example 1; i.e., $x_1 = [1,0]^\mathsf{T}$, $x_2 = [0,1]^\mathsf{T}$ and $x_3 = [1,1]^\mathsf{T}$. One can extend this set to $\mathbb{C}^{\otimes 2}$ by taking Kronecker products and restricting to $\mathrm{Sym}^2(\mathbb{C}^2)$ yields $x_1^{\otimes 2} = [1,0,0]^\mathsf{T}$, $x_2^{\otimes 2} = [0,0,1]^\mathsf{T}$, and $x_3^{\otimes 2} = [1,1,1]^\mathsf{T}$. Let $X^{(2)} = \{x_1^{\otimes 2}, x_2^{\otimes 2}, x_3^{\otimes 2}\}$. The polynomial p can be uniquely determined from its sample values at x_1, x_2 and x_3 because $c^{(1)} = p(x_1)$, $c^{(3)} = p(x_2)$, and $c^{(2)} = p(x_3) - p(x_2) - p(x_1)$. This means that $X^{(2)}$ is a frame for $\mathrm{Sym}^2(\mathbb{C}^2)$. The frame operator is

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

making the metric operator

$$\mathcal{F} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The minimum and maximum eigenvalues of $\mathcal{F}$ are .2679 and 3.7321, which are the optimal lower and upper frame bounds respectively. The metric operator for the dual frame is

$$\mathcal{F}^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

making the dual frame $\widetilde{x_1^{\otimes 2}} = \mathcal{F}^{-1}x_1^{\otimes 2} = [1,-1,0]^\mathsf{T}$, $\widetilde{x_2^{\otimes 2}} = \mathcal{F}^{-1}x_2^{\otimes 2} = [0,-1,1]^\mathsf{T}$, and $\widetilde{x_3^{\otimes 2}} = \mathcal{F}^{-1}x_3^{\otimes 2} = [0,1,0]^\mathsf{T}$.
The polynomial $p(u,v) = c^{(1)}u^2 + c^{(2)}uv + c^{(3)}v^2$ satisfies $p(x_1) = c^{(1)}, p(x_2) = c^{(3)}$, and $p(x_3) = c^{(1)} + c^{(2)} + c^{(3)}$. The coefficients of p can be obtained from its samples at x_1, x_2, and x_3 by the frame reconstruction formula for $\mathrm{Sym}^2(\mathbb{C}^2)$; i.e.,

$$[c^{(1)}, c^{(2)}, c^{(3)}]^\mathsf{T} = p(x_1)\widetilde{x_1^{\otimes 2}} + p(x_2)\widetilde{x_2^{\otimes 2}} + p(x_3)\widetilde{x_3^{\otimes 2}}.$$

Example 3. Consider now the frame for $\mathbb{C}^2$ formed by $x_1 = [1,0]^\mathsf{T}$, $x_2 = [2,0]^\mathsf{T}$, and $x_3 = [0,1]^\mathsf{T}$. In this case, reconstruction of $p(u,v) = c^{(1)}u^2 + c^{(2)}uv + c^{(3)}v^2$ from samples $p(x_1)$, $p(x_2)$, and $p(x_3)$ is not generally possible, even though the number of samples is the same as the dimension of $H_2(\mathbb{C}^2)$. This is because x_1 and x_2 are scalar multiples of each other and the corresponding vectors in $\mathrm{Sym}^2(\mathbb{C}^2)$, $\{[1,0,0]^\mathsf{T}, [2,0,0]^\mathsf{T}, [0,0,1]^\mathsf{T}\}$ do not constitute a frame for $\mathrm{Sym}^2(\mathbb{C}^2)$. This is an example where the tensor powers of a frame for V do not frame $\mathrm{Sym}^k(V)$, even though the number of vectors is adequate.

Example 4. Reconstruction of homogeneous polynomials in $H_3(\mathbb{C}^2)$ requires at least four points, since the dimension of $\mathrm{Sym}^3(\mathbb{C}^2)$ and hence that of $H_3(\mathbb{C}^2)$ is four. Taking the frame $X = \{x_1, x_2, x_3, x_4\} = \{[1,0]^\mathsf{T}, [0,1]^\mathsf{T}, [1,1]^\mathsf{T}, [1,-1]^\mathsf{T}\}$ for $\mathbb{C}^2$, computing Kronecker products and restricting to $\mathrm{Sym}^3(\mathbb{C}^2)$ yields

$$\begin{aligned} X^{(3)} &= \{x_1^{\otimes 3}, x_2^{\otimes 3}, x_3^{\otimes 3}, x_4^{\otimes 3}\} \\ &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}. \end{aligned}$$

A homogeneous polynomial of the form $p(u,v) = c^{(1)}u^3 + c^{(2)}u^2v + c^{(3)}uv^2 + c^{(4)}v^3$ can be reconstructed from its samples at these points as $c^{(1)} = p(1,0)$, $c^{(2)} = \frac{1}{2}(p(1,1) + p(1,-1) - 2p(1,0))$, $c^{(3)} = \frac{1}{2}(p(1,1) + p(1,-1) - 2p(1,0))$, and $c^{(4)} = p(0,1)$ so that $X^{(3)}$ constitutes a frame for $\mathrm{Sym}^3(\mathbb{C})^2$. The frame operator is

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

making the metric operator

$$\mathcal{F} = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \end{bmatrix}.$$

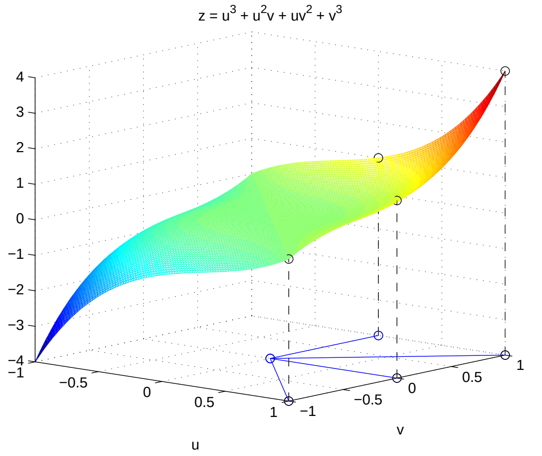

Figure 2: A homogeneous polynomial of degree three.

The optimal lower and upper frame bounds are $A = 0.4384$ and $B = 4.5616$. The metric operator of the dual frame is

$$\mathcal{F}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1.5 & 0 & -1 \\ -1 & 0 & 1.5 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

The dual frame is $\widetilde{x_1^{\otimes 3}} = \mathcal{F}^{-1}x_1^{\otimes 3} = [1, 0, -1, 0]^\mathsf{T}$, $\widetilde{x_2^{\otimes 3}} = \mathcal{F}^{-1}x_2^{\otimes 3} = [0, -1, 0, 1]^\mathsf{T}$, $\widetilde{x_3^{\otimes 3}} = \mathcal{F}^{-1}x_3^{\otimes 3} = [0, .5, .5, 0]^\mathsf{T}$, and $\widetilde{x_4^{\otimes 3}} = \mathcal{F}^{-1}x_4^{\otimes 3} = [0, -.5, .5, 0]^\mathsf{T}$. The coefficients of a degree-three polynomial $p(u, v) = c^{(1)}u^3 + c^{(2)}u^2v + c^{(3)}uv^2 + c^{(4)}v^3$ are given by

$$[c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}]^\mathsf{T} = p(x_1)\widetilde{x_1^{\otimes 3}} + p(x_2)\widetilde{x_2^{\otimes 3}} + p(x_3)\widetilde{x_3^{\otimes 3}} + p(x_4)\widetilde{x_4^{\otimes 3}}.$$

Such a polynomial and the sampling points are shown in Figure 2.

Example 5. As the degree k or the dimension n gets larger numerical issues arise in calculating the inverse of the metric operator in order to get the dual frame that is needed for the reconstruction [4]. Ideally, one would like to construct tight frames for $\mathrm{Sym}^k(\mathbb{C}^n)$. Since the upper and lower frame bounds determine the numerical merits of a particular frame, it is interesting to observe how starting with a fixed frame for $\mathbb{C}^2$ the frame bounds change as this frame is extended to frames for $\mathrm{Sym}^k(\mathbb{C}^2)$ as k increases. Taking the frame for $\mathbb{C}^2$ to be $\{[1, 0]^\mathsf{T}, [0, 1]^\mathsf{T}, [1, 1]^\mathsf{T}, [1, -1]^\mathsf{T}\}$, the frame bounds for $\mathbb{C}^2$, $\mathrm{Sym}^2(\mathbb{C}^2)$, and $\mathrm{Sym}^3(\mathbb{C}^2)$ are tabulated below.

Space	Optimal lower frame bound A	Optimal upper frame bound B	B/A
$\mathbb{C}^2$	3	3	1
$\mathrm{Sym}^2(\mathbb{C}^2)$	1	5	5
$\mathrm{Sym}^3(\mathbb{C}^2)$	.4384	4.5616	10.4

In this particular case it appears that the ratio B/A increases as k, the degree of the polynomial increases.

4. Conclusions and Future Work

It has been shown that homogeneous polynomials of degree k in n variables can be reconstructed from their samples at elements of a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$. Such a set can also be used to reconstruct n-variate homogeneous polynomials of all degrees ℓ where $1 \leqslant \ell < k$. In recent work [1], [9] conditions under which a smooth function can be approximated by homogeneous polynomials have been established. Combining these results to approximately reconstruct smooth functions from sampled data and a possible construction of tight frames for $\mathrm{Sym}^k(\mathbb{C}^n)$ will be given in a detailed version of this work. The metric operator and the grammian of a frame have the same non-zero eigenvalues. Also $\mathcal{G}^{\circ k}$, the grammian of $X^{(k)} = \{x_1^{\otimes k}, x_2^{\otimes k}, \ldots, x_m^{\otimes k}\}$, is the k-fold Hadamard product of $\mathcal{G}$, the grammian of $X = \{x_1, x_2, \ldots, x_m\}$. Relationship between the eigenvalues of $\mathcal{G}$ and $\mathcal{G}^{\circ k}$ ([2], [5], [6]) may be used to obtain information about the frame bounds for a frame for $\mathrm{Sym}^k(\mathbb{C}^n)$ which comes from a frame for $\mathbb{C}^n$, see Example 5.

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References:

[1] D. Benko and A. Kroó. A Weierstrass-type theorem for homogeneous polynomials. *Transactions of the American Mathematical Society*, 361(3):1645 – 1665, 2009.

[2] G. Cheng, X. Cheng, T. Huang, and T. Tam. Some bounds for the spectral radius of the Hadamard product of matrices. *Applied Mathematics E-Notes*, 5:202–209, 2005.

[3] S. Datta, S. D. Howard, and D. Cochran. Geometry of the Welch bounds. *IEEE Transactions on Information Theory*. In review.

[4] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.

[5] M. Fang. Bounds on eigenvalues of the Hadamard product and the Fan product of matrices. *Linear Algebra and its Applications*, 425:7–15, 2007.

[6] E. I. Im. Narrower eigenbounds for Hadamard products. *Linear Algebra and its Applications*, 264:141–144, 1997.

[7] I. Peng and S. Waldron. Signed frames and Hadamard products of Gram matrices. *Linear Algebra and its Applications*, 347:131–157, 2002.

[8] R. Shaw. *Linear algebra and group representations*, volume 2. Academic Press, 1983.

[9] P. Varjú. Approximation by homogeneous polynomials. *Constructive Approximation*, 26:317 – 337, 2007.

[10] A. Zakhor and G. Alvstad. Two-dimensional polynomial interpolation from nonuniform samples. *IEEE Transactions on Signal Processing*, 40(1):169 – 180, 1992.