Reproducing Kernel Structure and Sampling on Time-Warped Spaces With Application to Warped Wavelets

Shahrnaz Azizi, *Member, IEEE*,
Douglas Cochran, *Senior Member, IEEE*, and John N. McDonald

***Abstract*—Time-warped signal spaces have received recent attention in the research literature. Among the topics of particular interest are sampling of time-warped signals and signal analysis using warped analysis functions, including wavelets. This correspondence introduces a reproducing kernel (RK) structure for time-warped signal spaces that unifies multiple perspectives on sampling in such spaces.**

***Index Terms*—Nonuniform sampling, reproducing kernel (RK), sampling, time warping, wavelets.**

I. INTRODUCTION

Given a space $\mathcal{S}$ of functions $f: \mathbb{R} \to \mathbb{C}$ and an invertible continuous function $\gamma: \mathbb{R} \to \mathbb{R}$, the *time-warped function space* $\mathcal{S}_\gamma$ consists of all functions of the form $f_\gamma = f \circ \gamma$ (i.e., $f_\gamma(t) = f(\gamma(t))$ for all $t \in \mathbb{R}$). In this context, γ is called the *warping function*. In the case that $\mathcal{S}$ is the space $\mathcal{B}$ of Ω-bandlimited signals (i.e., functions of the form

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

with $0 < \Omega < \infty$ and $f \in L^2(\mathbb{R})$), a result of Clark *et al.* [1] shows that the space $\mathcal{B}_\gamma$ of time-warped band-limited signals admits the formula

$$f_\gamma(t) = \sum_n f_\gamma(\tau_n) \operatorname{sinc}\left[\frac{\Omega(\gamma(t) - nT)}{\pi}\right]$$

for reconstruction of f_γ from samples $f_\gamma(\tau_n) = f_\gamma(\gamma^{-1}(nT))$. In this expression, $T = \pi/\Omega$ is the so-called Nyquist interval for $\mathcal{B}$ and $\operatorname{sinc}(t) = \sin(\pi t)/\pi t$. The sampling times $\{\tau_n\}$ are generally nonuniformly spaced and $\mathcal{B}_\gamma$ contains signals that are not band-limited [2], [3], so Clark's result provides a means for reconstructing certain spaces of non-band-limited signals from nonuniformly spaced samples.

Also, it has been observed in [2] that for any signal space $\mathcal{S}$ that admits a reconstruction formula of the form

$$f(t) = \sum_n f(t_n) \phi_n(t) \tag{1}$$

there is a sampling theorem with reconstruction formula

$$f_\gamma(t) = \sum_n f_\gamma(\tau_n) \phi_n(\gamma(t)), \qquad \tau_n = \gamma^{-1}(t_n)$$

for time-warped signals in $\mathcal{S}_\gamma$. Thus, Clark's basic idea applies to time-warped signal spaces in addition to $\mathcal{B}_\gamma$.

In [4], sampling theory for time-warped signal spaces in which Kramer's generalized sampling theorem [5] applies were studied in view of the well-established relationship between sampling and the reproducing kernel (RK) [6], [7] in a reproducing kernel Hilbert space (RKHS). It was shown that such spaces admit RKHS structures, in view of which three perspectives apply to yield sampling theorems: 1) by applying Clark's method, 2) with the RKHS inner product, such spaces were seen to themselves admit Kramer's theorem in which Clark's perspective is subsumed, and 3) as a corollary to a standard result about sampling formulas in RKHS. The corresponding sampling formulas are shown to be equivalent.

This correspondence considers general time-warped signal spaces $\mathcal{S}_\gamma$ where $\mathcal{S}$ is an RKHS in which a reconstruction formula of the form (1) holds. When endowed with the appropriate inner product, $\mathcal{S}_\gamma$ is shown to admit an RK structure and thus become an RKHS. The corresponding sampling theorem is, however, identical to the one obtained by applying Clark's method.

Ongoing development of sampling theorems for wavelet subspaces [8]–[11] has followed from [8] where it was shown that, under weak hypotheses, multiresolution subspaces are RKHSs. This motivates the application of the results developed to obtain sampling theorems in warped wavelet subspaces that are of interest in signal processing [12]–[14].

The rest of this correspondence is organized as follows. In Section II, first the RK structure for time-warped signals spaces is introduced. Then it is shown that the sampling basis obtained by applying Clark's method arises from the RK and hence the corresponding sampling theorem of RK is identical to the one obtained by applying Clark's method. Sampling theorems in warped wavelet subspaces are explained in Section III. This is followed by a summary and conclusions in Section IV.

II. RKHS STRUCTURE AND SAMPLING IN TIME-WARPED SPACES

Recall that an RK on a Hilbert space $\mathcal{H}$ of complex-valued functions on $\mathbb{R}$ is a function $k: \mathbb{R}^2 \to \mathbb{C}$ such that $k(\cdot, x) \in \mathcal{H}$ for each real x and $f(x) = \langle f, k(\cdot, x)\rangle$ for every $x \in \mathbb{R}$ and $f \in \mathcal{H}$. If an RK exists for $\mathcal{H}$, then it is unique.

Let $\mathcal{S}$ be an RKHS with inner product $\langle \cdot, \cdot \rangle$. It is proven in [7] that a sampling basis $\{\phi_n\}$ of $\mathcal{S}$ yields a reconstruction formula of the form (1) for a sampling set $\{t_n\}$ if and only if its biorthogonal[1] basis $\{\tilde{\phi}_n\}$ is given by

$$\tilde{\phi}_n(x) = \left\langle \tilde{\phi}_n, \phi_n \right\rangle k(t_n, x) \tag{2}$$

where k is the RK which can be given by

$$k(t, x) = \sum \left\langle \overline{\tilde{\phi}_n, \phi_n} \right\rangle^{-1} \phi_n(t) \overline{\tilde{\phi}_n(x)}. \tag{3}$$

Now consider the warped space $\mathcal{S}_\gamma$ with an invertible continuous function $\gamma: \mathbb{R} \to \mathbb{R}$ and define an inner product $\langle \cdot, \cdot \rangle_\gamma$ in $\mathcal{S}_\gamma$ by

$$\langle f_\gamma, g_\gamma \rangle_\gamma = \langle f, g \rangle \tag{4}$$

[1]Recall that two sets $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$ are biorthogonal if $\langle \phi_n, \tilde{\phi}_m \rangle = \delta_{nm} \langle \phi_n, \tilde{\phi}_m \rangle$.

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S. Azizi was with Arizona State University, Tempe, AZ 85287 USA. She is now with the Mobile Communication Division, Intel Corporation, San Diego, CA 92128 USA (e-mail: shahrnaz.azizi @intel.com).

D. Cochran was with Arizona State University, Tempe, AZ 85287 USA. He is now with DARPA/DSO, Arlington, VA, 22203-1714 USA (e-mail: dcochran @darpa.mil).

J. N. McDonald is with the Department of Mathematics, Arizona State University, Tempe, AZ 85287 USA (e-mail: mcdonald@math.la.asu.edu).

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and

$$k_\gamma(t, x) = k(\gamma(t), \gamma(x)).$$

Then, using (3) yields

$$k_\gamma(t, x) = \sum \left\langle \overline{\tilde{\phi}_n, \phi_n} \right\rangle^{-1} \phi_n(\gamma(t)) \overline{\tilde{\phi}_n(\gamma(x))}$$

and for a fixed $x \in \mathbb{R}$

$$\langle f_\gamma, k_\gamma(\cdot, x)\rangle_\gamma = \left\langle f, \sum \left\langle \overline{\tilde{\phi}_n, \phi_n} \right\rangle^{-1} \phi_n(\cdot)\overline{\tilde{\phi}_n(\gamma(x))} \right\rangle.$$

Applying (3) again results in

$$\langle f_\gamma, k_\gamma(\cdot, x)\rangle_\gamma = f(\gamma(x)) = f_\gamma(x).$$

Therefore, k_γ is the RK for $\mathcal{S}_\gamma$.

Clark's observation shows that the warped space $\mathcal{S}_\gamma$ admits a reconstruction formula with sampling times $\tau_n = \gamma^{-1}(t_n)$ and interpolation functions $\{v_n = \phi_n \circ \gamma\}$ obtained by warping original interpolation functions $\{\phi_n\}$. Using the RKHS structure on $\mathcal{S}_\gamma$ defined above allows this to be deduced as a corollary to a standard result about sampling formulas in RKHS. To see this, consider k_γ evaluated at the points τ_n

$$\begin{aligned} k_\gamma(\tau_n, x) &= \sum \left\langle \overline{\tilde{\phi}_n, \phi_n} \right\rangle^{-1} \phi_n(\gamma(\tau_n)) \overline{\tilde{\phi}_n(\gamma(x))} \\ &= \sum \left\langle \overline{\tilde{\phi}_n, \phi_n} \right\rangle^{-1} \phi_n(t_n) \overline{\tilde{\phi}_n(\gamma(x))} \\ &= k(t_n, \gamma(x)) \\ &= \left\langle \tilde{\phi}_n, \phi_n \right\rangle^{-1} \tilde{\phi}_n(\gamma(x)) \end{aligned}$$

where the last equation is obtained by applying (2) in $\mathcal{S}$. But $\tilde{\phi}_n(\gamma(x))$ is biorthogonal to $\{v_n\}$ because

$$\left\langle \tilde{\phi}_n(\gamma(\cdot)), v_n \right\rangle_\gamma = \left\langle \tilde{\phi}_n, \phi_n \right\rangle$$

by the inner product defined in (4). Therefore, the biorthogonal basis of the sampling basis $\{v_n\}$ arises from the RK, and (2) is satisfied in $\mathcal{S}_\gamma$.

III. Application to Time-Warped Wavelet Subspaces

As noted above, recent interest in sampling theorems for wavelet subspaces has grown from a result of Walter [8]. Under relatively weak conditions on a real-valued scaling function φ that generates a multiresolution analysis $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, each $\mathcal{V}_m$ was shown to be an RKHS with RK $q_m: \mathbb{R}^2 \to \mathbb{R}$ of the form

$$q_m(t, s) = 2^m \sum \varphi(2^m t - n)\varphi(2^m s - n).$$

It follows that each $f \in \mathcal{V}_0$ admits a (uniformly convergent) reconstruction from samples $\{f(n)\}_{n\in\mathbb{Z}}$ by the formula

$$f(t) = \sum f(n) p_0(t - n), \qquad t \in \mathbb{R}$$

where $\{p_0(\cdot - n)\}_{n\in\mathbb{Z}}$ is the basis of $\mathcal{V}_0$ biorthogonal to $\{q_0(\cdot, n)\}$. The obvious extension gives a sampling theorem in each $\mathcal{V}_m$.

Consider now the warped multiresolution subspace $\mathcal{V}_{0\gamma}$. As discussed above, $\mathcal{V}_{0\gamma}$ is itself an RKHS with RK $k_\gamma(t, s) = q_0(\gamma(t), \gamma(s))$ and thus admits a corresponding sampling theorem. The reconstruction formula is

$$f_\gamma(t) = \sum f_\gamma(\tau_n) p_0(\gamma(t) - n), \qquad \tau_n = \gamma^{-1}(n), \ t \in \mathbb{R}$$

i.e., exactly the one given by Clark's observation.

IV. Conclusion

Based on the preceding section, one may ask whether $\mathcal{V}_m$ remains a subspace of $L^2(\mathbb{R})$ after warping. This question is addressed in [15], where the answer is shown to be "yes" if γ is smooth and monotone increasing with $\frac{d}{dt}(\gamma^{-1})$ uniformly bounded.

This correspondence has shown that warping an RKHS in which a reconstruction formula from samples holds yields in a new space that itself admits an RKHS structure if the inner product is chosen appropriately. Introduction of the RKHS structure in warped spaces allows the use of standard results on sampling in RKHS to obtain sampling theorems for such spaces, including those generated by Clark's method. Coupled with Walter's observation that wavelet subspaces often have RKs, the main result of this correspondence gives a sampling theorem for certain warped wavelet subspaces as a direct corollary.

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