Asymptotic Analysis of the Generalized Coherence Estimate

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***Abstract*—The generalized coherence (GC) estimate has shown promise as a multiple-channel detection statistic, but analysis of its performance in this role has been hampered because its probability density function is difficult to evaluate under signal absent hypotheses and is unknown under signal-present hypotheses. This paper presents an asymptotic analysis of the GC estimate that provides tractable closed-form expressions for the density of the GC estimate under useful signal-absent and signal-present hypotheses. These expressions are valid as the number of data samples employed in the estimate approaches infinity. Theoretical predictions of detection performance based on these expressions are shown to match closely results obtained by Monte Carlo simulation, even when relatively small numbers of samples are used in the GC estimate. These results are used to further examine the merits of the GC estimate as a multiple-channel detection statistic.**

***Index Terms*—Correlators, multisensor systems, radar detection, signal detection, sonar detection.**

I. Introduction

THE generalized coherence (GC) estimate and its use as a statistic for detection of a common unknown signal component in multiple noisy channels were introduced in [1]. The GC estimate was observed to generalize the magnitude-squared coherence (MSC) estimate, which is widely used in two-channel detection and time-delay estimation [2]. As with the MSC estimate, the utility of the GC estimate as a detection statistic arises because its probability distribution is known under the signal-absent (H_0) assumptions that all channels contain independent complex white Gaussian noise. This knowledge allows detection thresholds corresponding to desired false-alarm probabilities to be explicitly calculated, thereby providing a foundation for a constant false alarm rate (CFAR) detector. Recent work on invariance properties of the GC estimate [3] has generalized earlier results for the MSC estimate [4], [5] by establishing that the distribution of the GC estimate is invariant to the distribution of one channel, provided the remaining channels contain only white Gaussian noise and that all channels are independent. It has recently been observed that the GC estimate is the generalized likelihood ratio test (GLRT) statistic to determine if the components of a complex Gaussian random vector are independent and that the GC estimate arises as the test statistic in the uniformly most powerful invariant matched subspace detector for a class of multiple-channel detection problems [6], [7].

A significant limitation encountered in application of the GC estimate is that its distribution within the framework of a useful signal-present (H_1) model is not known except in the two-channel (i.e., MSC) case. Consequently, all performance data for GC-based detectors using three or more channels has come from simulations. Another problem is that the distribution of the GC estimate under H_0 is given in [1] by a recursion formula that is intractable for more than a few channels. This paper addresses both of these issues by developing an asymptotic analysis of the GC estimate. This analysis provides closed-form expressions for the distribution of the estimate under H_0 as well as under an additive H_1 model that are valid as the number of data samples used in the estimate approaches infinity. By comparison with simulations, these expressions are shown to provide accurate predictions of detector performance, even when relatively small numbers of samples are used to form the GC estimate. Moreover, the tractability of the asymptotic H_0 distribution means detection thresholds can be easily computed for large numbers of channels. The availability of a distribution under H_1 allows performance comparison between GC-based detectors and multiple-channel detectors based on the multiple coherence (MC) estimate [8].

The remainder of this paper is organized as follows. The next section describes the multiple-channel detection scenario and defines generalized coherence and the GC estimate. In Section III, asymptotic expressions are developed for the H_0 distribution and for the non-null distribution under an explicit set of H_1 assumptions. Section IV uses these results to present an asymptotic performance analysis for GC-based detectors. Receiver operating characteristic (ROC) curves derived from theory are compared with empirically determined ROC curves for various scenarios. The performance of GC-based and MC-based detectors is also compared in Section IV.

II. Generalized Coherence

Suppose that the M-channel detector is to operate in a scenario

$$\begin{aligned} H_0&:\ z_k(\cdot) = n_k(\cdot)\\ H_1&:\ z_k(\cdot) = s(\cdot) + n_k(\cdot) \end{aligned}$$

where $s(\cdot)$ denotes a common signal with spectral density $S_s(\cdot)$, and the noise $n_k, k = 1, \cdots, M$ on each channel is independent and complex Gaussian with spectral density $S_{n_k}(\cdot)$. For a white

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signal in additive white noise, the vectors $\mathbf{z}$ obtained by sampling the processes $z_k(\cdot)$ can be modeled as independent realizations of a complex random M-vector $\mathbf{Z} = [Z_1 \quad \cdots \quad Z_M]^T$.

Suppose that N independent observations $\mathbf{z}_\alpha$ of the random vector $\mathbf{Z}$ are available. The GC estimate, which is introduced in [1], is defined as

$$\hat{\gamma}^2(A) \hat{=} 1 - V = 1 - \frac{\det A}{\prod_{k=1}^{M} A_{kk}} \tag{1}$$

where

$$A \hat{=} \frac{1}{N} \sum_{\alpha=1}^{N} \mathbf{z}_\alpha \mathbf{z}_\alpha^H \tag{2}$$

$\mathbf{z}^H$ denotes the complex conjugate transpose of $\mathbf{z}$, and A_{kk} is the kth diagonal element of the matrix A. Note that the matrix A has a complex Wishart distribution, which was introduced in [9] as

$$f_W(A) = \frac{(\det A)^{N-M}}{I_{N,M}(\Sigma)} \exp[-\mathrm{tr}(\Sigma^{-1}A)]$$

with

$$I_{N,M}(\Sigma) = \pi^{(1/2)M(M-1)}\Gamma(N)\cdots\Gamma(N-M+1)(\det\Sigma)^N.$$

The moments of V as given in (1) can be calculated as

$$E[V^h] = \int V^h(A) f_W(A)\, dA$$

where the integration is performed over all the elements of A. Applying the fact that under H_0 all the components of $\mathbf{Z}$ are independent yields

$$\begin{aligned} E[V^h|H_0] &= \prod_{k=1}^{M} \frac{\Gamma(N+h+1-k)}{\Gamma(N+1-k)} \left\{ \frac{\Gamma(N)}{\Gamma(N+h)} \right\}^M \\ &= \prod_{k=1}^{M-1} \frac{\Gamma(N+h-k)}{\Gamma(N-k)} \left\{ \frac{\Gamma(N)}{\Gamma(N+h)} \right\}^{M-1} \end{aligned} \tag{3}$$

and the hth moment of the GC estimate $\hat{\gamma}^2$ under H_0 is

$$E[(\hat{\gamma}^2)^h] = E[(1-V)^h]. \tag{4}$$

Prior work has focused on the GC estimate without significant attention to the underlying entity being estimated. In analogy with the well-known magnitude-squared coherence coefficient, the generalized coherence coefficient of a complex random vector $\mathbf{Z}$ with covariance matrix Σ may be defined as

$$\gamma^2 \hat{=} 1 - \frac{\det \Sigma}{\prod_{k=1}^{M} \Sigma_{kk}}. \tag{5}$$

It can be shown that the GC estimate (1) provides a consistent estimate of γ^2.

The GC coefficient γ^2 is a measure of the degree of correlation of the components of $\mathbf{Z}$ and has the properties $0 \le \gamma^2 \le 1$ for all $\mathbf{Z}$, $\gamma^2 = 0$ if and only if all components of $\mathbf{Z}$ are uncorrelated and $\gamma^2 = 1$ if any two components are perfectly correlated. The GC coefficient can be expressed in terms of the signal-to-noise ratios (SNRs) on the M channels as follows. The SNR on the kth channel at frequency ω is defined as

$$\mathrm{SNR}_k(\omega) = \frac{S_s(\omega)}{S_{n_k}(\omega).}$$

For a white signal in white noise, the spectral densities (and therefore the SNRs) are independent of frequency. Substituting the expression for the SNR into (5), it is possible to express the GC coefficient in terms of the SNRs on the channels as

$$\gamma_M^2(\mathrm{SNR}_1, \cdots, \mathrm{SNR}_M) = \frac{\sum_{i=2}^{M} C(i, \mathcal{S})}{\prod_{i=1}^{M}(1+\mathrm{SNR}_i)} \tag{6}$$

where $\mathcal{S}$ denotes the set $\mathcal{S} = \{\mathrm{SNR}_1, \mathrm{SNR}_2, \cdots, \mathrm{SNR}_M\}$, and $C(i, \mathcal{S})$ denotes the sum of all the i-tuples from $\mathcal{S}$. For example, $C(2, \mathcal{S}) = \mathrm{SNR}_1\,\mathrm{SNR}_2 + \mathrm{SNR}_1\,\mathrm{SNR}_3 + \cdots + \mathrm{SNR}_{M-1}\,\mathrm{SNR}_M$.

For equal SNRs on all channels, the GC coefficient for M channels can be written in terms of the SNRs as

$$\gamma_M^2(\mathrm{SNR}) = \frac{\sum_{i=0}^{M-2} \binom{M}{i} \mathrm{SNR}^{M-i}}{(1+\mathrm{SNR})^M}.$$

III. Asymptotic Distribution

In this section, asymptotic distributions of the GC estimate are derived under the H_0 assumption that all channels contain independent white Gaussian noise and under the H_1 assumption of a common white Gaussian signal in additive independent white Gaussian noise.

A. Asymptotic H_0 Distribution

In [1], the H_0 distribution of the GC estimate was developed by a geometrical argument. A closed-form expression was presented for $M = 3$ channels, and a recursion formula was given to obtain the distribution for more than three channels. However, the recursion formula is not easy to evaluate, and thus far, no closed-form expressions of the distribution are known for $M > 3$ channels.

The derivation of the asymptotic H_0 distribution follows an approach introduced by Box [10], which provides a general asymptotic expansion of the distribution of a random variable W whose hth moment is of the form

$$E[W^h] = K_1 \left(\frac{\prod_{j=1}^{b} y_j^{y_j}}{\prod_{k=1}^{a} x_k^{x_k}} \right)^h \frac{\prod_{k=1}^{a} \Gamma[x_k(1+h)+\xi_k]}{\prod_{j=1}^{b} \Gamma[y_j(1+h)+\eta_j]} \tag{7}$$

for $h = 0, 1, 2, \cdots$. In (7), K_1 is a constant such that $E[W^0] = 1$, and it is required that $\sum_{k=1}^{a} x_k = \sum_{j=1}^{b} y_j$.

In the previous section, the moments of the GC estimate were derived under the H_0 assumption. The moments of V are given in (3), and the moments of $\lambda = V^N$ are

$$E[\lambda^h] = K_2 \prod_{k=1}^{M} \frac{\Gamma[N(1+h)+1-k]}{\Gamma[N(1+h)]} \tag{8}$$

where the constant K_2 contains all terms that do not depend on h. Comparison of (8) with (7) yields $a = b = M, x_k = N, y_j = N, \xi_k = 1-k$, and $\eta_j = 0$. From (3), it follows that $E[\lambda^0] = 1$, and Box's theory can be applied to λ. The distribution of the GC estimate $\hat{\gamma}^2 = 1 - V = 1 - \lambda^{1/N}$ follows immediately from the distribution of V.

Consider $Y = -\rho \log \lambda = -N\rho \log V$, where ρ will be determined later. Applying Box's approach, the asymptotic distribution is found by calculating the characteristic function of $\log Y$ and using an expansion formula for the gamma function in terms of Bernoulli polynomials. Details of this expansion can be found in [11]. The distribution of Y is

$$\begin{aligned}\Pr(&-N\rho \log V < z)\\ &= \Pr(\chi_f^2 \le z) + \omega_1[\Pr(\chi_{f+2}^2 \le z) - \Pr(\chi_f^2 \le z)]\\ &\quad + \omega_2[\Pr(\chi_{f+4}^2 \le z) - \Pr(\chi_f^2 \le z)]\\ &\quad + \frac{\omega_1^2}{2}[\Pr(\chi_{f+4}^2 \le z) - 2\Pr(\chi_{f+2}^2 \le z) + \Pr(\chi_f^2 \le z)]\\ &\quad + \mathcal{O}\left(N^{-3}\right)\end{aligned} \tag{9}$$

where χ_f^2 is the chi-squared distribution with f degrees of freedom

$$\omega_r = \left(\frac{(-1)^{r+1}}{r(r+1)N^r\rho^r}\right) \times \left[\sum_{k=1}^{M} B_{r+1}(N(1-\rho)+1-k) - B_{r+1}(N(1-\rho))\right]$$

B_r is the Bernoulli polynomial of degree r and order unity [12], and

$$f = -2\sum_{k=1}^{M} \xi_k = -2\sum_{k=1}^{M} 1-k = M(M-1).$$

Box suggests calculating ρ such that $\omega_1 = 0$. Using the relation for Bernoulli polynomials that $B_2(h) = h^2 - h + (1/6)$, ω_1 can be expanded as

$$\begin{aligned}\omega_1 = \frac{1}{2\rho N}\sum_{k=1}^{M}\{&((1-\rho)N + 1 - k)^2 - (1-\rho)N\\ &+ 1 - k - (1-\rho)^2N^2 - (1-\rho)N\}\end{aligned}$$

and

$$\rho = 1 - \frac{\sum_{k=1}^{M}(1-k)^2}{NM(M-1)} - \frac{1}{2N}.$$

With $\omega_1 = 0$, the distribution simplifies to

$$\begin{aligned}\Pr(&-N\rho \log V < z)\\ &= \Pr(\chi_f^2 \le z) + \omega_2[\Pr(\chi_{f+4}^2 \le z) - \Pr(\chi_f^2 \le z)]\\ &\quad + \omega_3[\Pr(\chi_{f+6}^2 \le z) - \Pr(\chi_f^2 \le z)] + \mathcal{O}(N^{-4})\end{aligned} \tag{10}$$

where

$$\omega_2 = -\frac{1}{6N^2\rho^2}\left[\frac{3}{2}\gamma^2 f + \sum_{k=1}^{M} B_3(1-k) - B_3(0)\right]$$

$$\omega_3 = \frac{1}{12N^3\rho^3}\left[-2\gamma^3 f - 24\gamma N^2\rho^2\omega_2 + \sum_{k=1}^{M} B_4(1-k) - B_4(0)\right]$$

with

$$\gamma = \frac{1}{f}\left[\sum_{k=1}^{M} B_2(1-k) - B_2(0)\right].$$

For most practical applications, it is sufficient if the series is terminated after three terms. The distribution of the GC estimate can be obtained in the following manner. Let

$$\Pr(\hat{\gamma}^2 < z) = \Pr(1 - V < z) = 1 - \Pr(V < 1 - z)$$

and therefore

$$\Pr(-N\rho \log \hat{\gamma}^2 < z) = 1 - \Pr(-N\rho \log V < 1 - z).$$

Finally, the distribution of $\hat{\gamma}^2$ is given as

$$\begin{aligned}\Pr(&\hat{\gamma}^2 < z)\\ &= 1 - \Pr(\chi_f^2 \le 1 - z')\\ &\quad - \omega_2[\Pr(\chi_{f+4}^2 \le 1 - z') - \Pr(\chi_f^2 \le 1 - z')]\\ &\quad - \omega_3[\Pr(\chi_{f+6}^2 \le 1 - z') - \Pr(\chi_f^2 \le 1 - z')]\\ &\quad + \mathcal{O}\left(N^{-4}\right)\end{aligned} \tag{11}$$

with $z' = -N\rho \log z$.

B. Asymptotic H_1 Distribution

The goal of this section is to establish that for a complex zero-mean white Gaussian signal in complex zero-mean white Gaussian noise, the conditional distribution of $\hat{\gamma}^2$, given $\gamma^2 > 0$, is asymptotically normal with mean $E[\hat{\gamma}^2|\gamma^2] = \gamma^2$ and variance

$$\text{var}(\hat{\gamma}^2|\gamma^2, R) = \frac{(1-\gamma^2)^2(\text{tr}(R^2) - M)}{N} \tag{12}$$

where R denotes the matrix of correlation coefficients between the channels. Under these conditions, $\mathbf{Z}$ is a zero-mean complex Gaussian random M-vector [9] with covariance matrix $\Sigma = E[\mathbf{Z}\mathbf{Z}^H]$.

It clearly suffices to establish that V, as defined in (1), is asymptotically normal with the appropriate mean and variance. To achieve this, V is shown in the following paragraphs to be an affine function of a complex random vector Φ that is asymptotically normal.

Denote $T = \sqrt{N}(A - \Sigma)$. Calculating the characteristic function of T and applying the central limit theorem for complex random variables [13] shows that the asymptotic distribution of T is complex Gaussian. Its mean is $E[T] = \sqrt{N}(E[A] - \Sigma) = \sqrt{N}(\Sigma - \Sigma) = 0$. To calculate the covariances of the elements of T, first observe that the covariances of the elements of A are given as

$$\begin{aligned}\text{cov}(A_{ij}, A_{kl}) &= E[(A_{ij} - E[A_{ij}])(A_{kl} - E[A_{kl}])^*] \\ &= \frac{1}{N}\Sigma_{ik}\Sigma_{lj} \end{aligned} \tag{13}$$

which is obtained by expanding the product and substituting the fourth moment of a zero-mean complex Gaussian random vector, which is given in [13] as

$$E[Z_i Z_j^* Z_k^* Z_l] = \Sigma_{ij}\Sigma_{lk} + \Sigma_{ik}\Sigma_{lj}.$$

Now, since

$$\begin{aligned}\text{cov}(T_{ij}, T_{kl}) &= NE[(A_{ij} - \Sigma_{ij})(A_{kl} - \Sigma_{kl})^*] \\ &= NE[A_{ij}A_{kl}^*] - N\Sigma_{ij}\Sigma_{kl}^*\end{aligned}$$

substituting (13) yields

$$\text{cov}(T_{ij}, T_{kl}) = \Sigma_{ik}\Sigma_{lj}. \tag{14}$$

Define Φ to be the vector obtained by arranging the elements of T in row-major order. Then, Φ is asymptotically zero-mean Gaussian, as desired, and its covariance matrix Ψ is determined by (14).

It remains to be shown that in the limit $N \to \infty$, V is an affine function of Φ. The Taylor expansion of $V = f(A)$ at $A = \Sigma$ is

$$f(A) = f(\Sigma) + f'(A)|_{A=\Sigma}(A - \Sigma) + \cdots$$

In this expression, the notation $(A - \Sigma)$ is understood to represent a column vector formed from the matrix elements in row-major order, and $f'(A)$ is a row vector of partial derivatives of f with respect to the elements of A, again taken in row-major order. Assuming the higher order terms are negligible for sufficiently large N yields the desired expression for V as an affine function of Φ

$$V = f(A) = f(\Sigma) + \Gamma\Phi \tag{15}$$

where $\Gamma = f'(A)|_{A=\Sigma}$.

The asymptotic mean of $\hat{\gamma}^2 = 1 - V$ can be calculated by taking the mean on both sides of (15) as

$$E(\hat{\gamma}^2|\gamma^2) = E[1 - V] = 1 - \frac{\det \Sigma}{\prod_{k=1}^{M} \Sigma_{kk}} = \gamma^2.$$

Standard results on Gaussian variates suggest that the asymptotic variance of V can be obtained using (15) and the covariance

TABLE I
COMPARISON OF DETECTION THRESHOLDS OBTAINED FROM THE EXACT DISTRIBUTION (FIRST VALUE) AND THE ASYMPTOTIC DISTRIBUTION (SECOND VALUE) FOR A THREE-CHANNEL GC-BASED DETECTOR CORRESPONDING TO CERTAIN PROBABILITIES OF FALSE ALARM AND DIFFERENT SAMPLE SIZES N

P_{fa}	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	.1593688	.0814292	.0411484	.0206823	.0103682	.0051908
	.1593688	.0814292	.0411484	.0206824	.0103682	.0051908
10^{-2}	.2398268	.1255405	.0642100	.0324690	.0163260	.0081859
	.2398267	.1255405	.0642100	.0324690	.0163260	.0081859
10^{-3}	.3067238	.1640643	.0848354	.0431347	.0217487	.0109199
	.3067236	.1640643	.0848354	.0431348	.0217487	.0109199
10^{-4}	.3651875	.1993141	.1041324	.0532235	.0269059	.0135271
	.3651871	.1993141	.1041324	.0532235	.0269059	.0135271
10^{-5}	.4173248	.2321731	.1225106	.0629338	.0318958	.0160563
	.4173241	.2321731	.1225106	.0629338	.0318958	.0160563
10^{-6}	.4643189	.2631002	.1401745	.0723635	.0367664	.0185313
	.4643178	.2631002	.1401745	.0723635	.0367664	.0185313

TABLE II
DETECTION THRESHOLDS OBTAINED FROM THE ASYMPTOTIC DISTRIBUTION CORRESPONDING TO CERTAIN PROBABILITIES OF FALSE ALARM AND DIFFERENT SAMPLE SIZES N FOR A GC-BASED DETECTOR WITH $M = 4, 5, 10, 20$ CHANNELS

M	P_{fa}	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
4	10^{-1}	.2635962	.1382728	.0707870	.0358101	.0180097	.0090310
	10^{-2}	.3511344	.1896914	.0985641	.0502355	.0253591	.0127403
	10^{-3}	.4190115	.2320601	.1221296	.0626500	.0317290	.0159664
	10^{-4}	.4757717	.2694799	.1434956	.0740514	.0376164	.0189578
	10^{-5}	.5247820	.3034975	.1634047	.0848048	.0432026	.0218045
5	10^{-1}	.3775851	.2048288	.1066297	.0543948	.0274706	.0138040
	10^{-2}	.4658432	.2614463	.1385042	.0712826	.0361599	.0182109
	10^{-3}	.5307205	.3062120	.1645944	.0853414	.0434541	.0219257
	10^{-4}	.5830433	.3446958	.1877143	.0979855	.0500625	.0253035
	10^{-5}	.6269817	.3789831	.2089002	.1097320	.0562436	.0284736
10	10^{-1}	.8520741	.5904695	.3512559	.1919720	.1003989	.0513459
	10^{-2}	.8898490	.6430755	.3930533	.2180461	.1149263	.0590091
	10^{-3}	.9127701	.6798620	.4242003	.2380736	.1262511	.0650268
	10^{-4}	.9288338	.7088418	.4500599	.2551226	.1360103	.0702441
	10^{-5}	.9408675	.7329306	.4725828	.2703058	.1447966	.0749666
20	10^{-1}	.9998074	.9745523	.8210222	.5661679	.3374438	.1848596
	10^{-2}	.9998956	.9806963	.8427971	.5926321	.3576818	.1973186
	10^{-3}	.9999347	.9843594	.8575877	.6117010	.3726807	.2066803
	10^{-4}	.9999563	.9869175	.8690462	.6271863	.3851379	.2145415
	10^{-5}	.9999694	.9888411	.8784724	.6404565	.3960234	.2214768

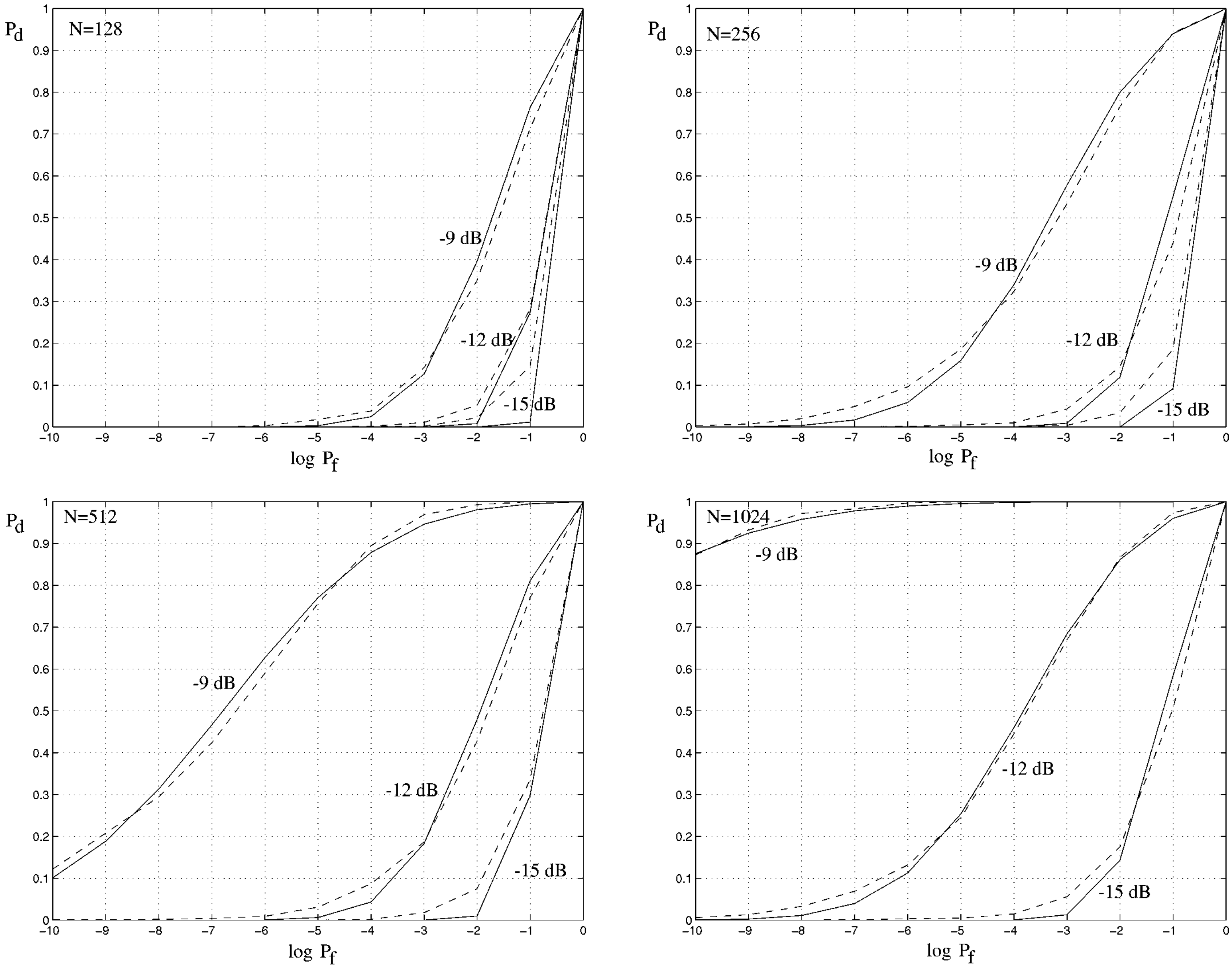

Fig. 1. Comparison of empirically determined (dashed lines) to asymptotic ROC curves (solid lines) of a three-channel GC estimate with equal SNR on all channels. The empirical distributions were obtained using 1000-trial Monte Carlo simulations. Curves are given for -9 dB, -12 dB, and -15 dB and different vector lengths N.

matrix Ψ of Φ calculated above. Following [14], the computation is accomplished using the total differential $df(A)$ at $A = \Sigma$:

$$\begin{aligned}\mathrm{var}(V) &= E[\{df(A)|_{A=\Sigma}\}^2] \\ &= E\left[\left\{\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial f(A)}{\partial A_{ij}}\,dA_{ij}\bigg|_{A=\Sigma}\right\}^2\right] \\ &= E\left[\left\{\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial f(A)}{\partial A_{ij}}\bigg|_{A=\Sigma}(A_{ij}-\Sigma_{ij})\right\}^2\right].\end{aligned}$$

Substituting $\gamma^2 = 1 - f(\Sigma)$ and $\hat{\gamma}^2 = 1 - V$ allows this equation to be simplified into the expression for the asymptotic variance of the GC estimate given in (12). For equal SNR on all channels, the variance simplifies to

$$\mathrm{var}(\hat{\gamma}^2|\gamma^2) = \frac{1}{N}(1-\gamma^2)^2 M(M-1)\rho^2$$

where ρ denotes the correlation coefficient between any two channels.

The preceding results establish that the conditional distribution of the GC estimate, given true GC value $\gamma^2 > 0$, is asymptotically Gaussian with mean γ^2 and variance given in (12).

IV. Nonparametric Detection

This section presents several results for a multiple-channel nonparametric detector based on the GC estimate as introduced in [1]. The GC-based detector distinguishes between the hypotheses H_0 (all channels contain independent zero-mean white Gaussian noise) and its complement H_1. It does not assume an explicit signal model but relies on the ability of the GC estimate to discern deviations from the H_0 hypothesis.

Based on the results developed in the previous sections, an asymptotic performance analysis for the GC detector is presented and it is evaluated in different detection scenarios.

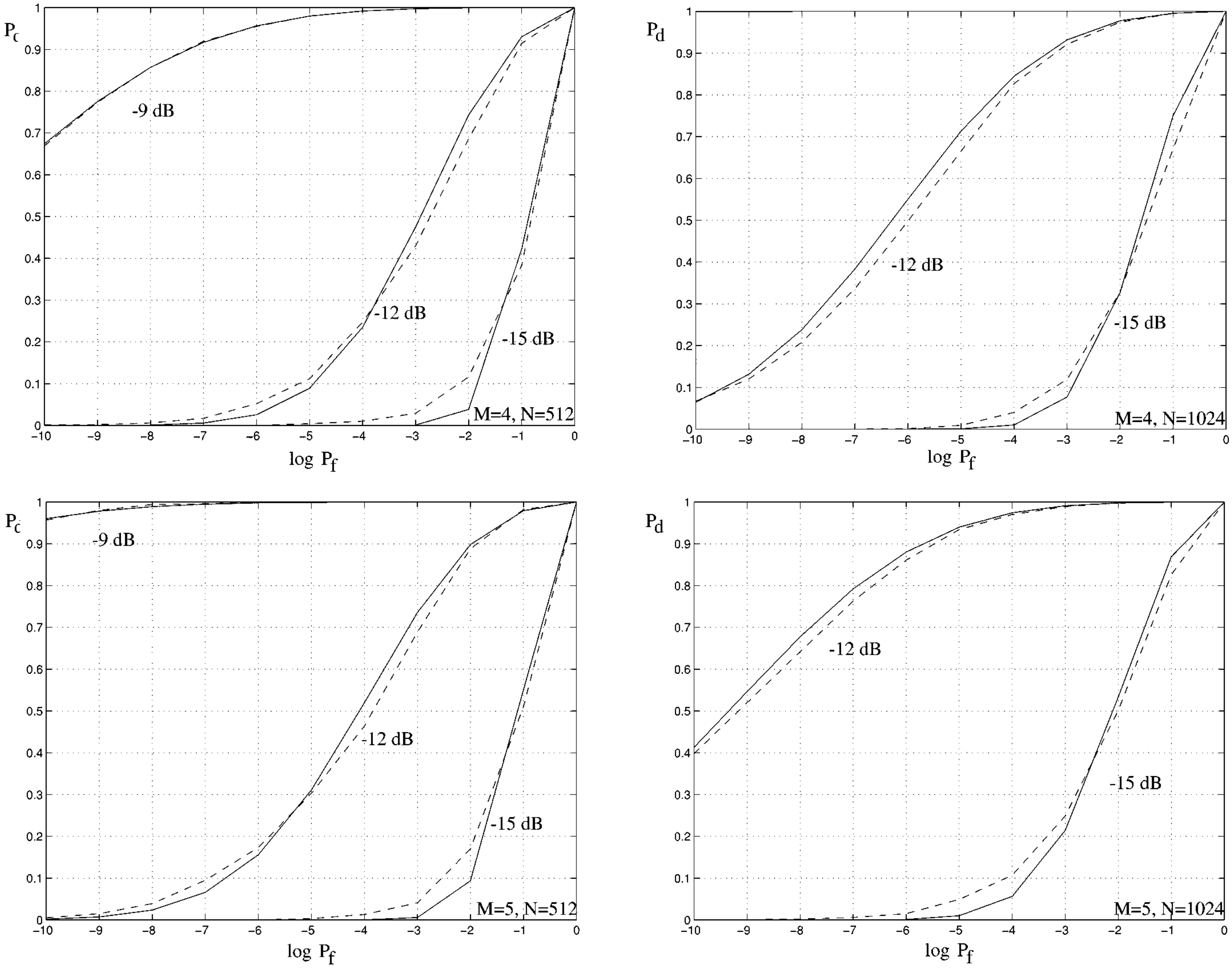

Fig. 2. Comparison of ROC curves for a M-channel GC detector obtained from the asymptotic distribution (solid lines) and from a Monte Carlo simulation (dashed lines) for equal SNR on all channels. Curves are given for -9 dB, -12 dB, and -15 dB for $M = 4$ (above) and $M = 5$ (below) channels and for samples sizes of $N = 512$ (left) and $N = 1024$ (right). Note that the -9 dB curves fall outside the axis limits in the $N = 1024$ graphs.

A. Computation of Detection Thresholds

Knowledge of the distribution under the H_0 hypothesis allows the calculation of detection thresholds for a constant false alarm rate (CFAR) receiver.

In Table II, thresholds obtained from the asymptotic H_0 distribution are given for GC estimates with $M = 4$, 5, 10, and 20 channels. For $M > 3$, no exact thresholds are known, and the thresholds obtained from the asymptotic distribution were verified by simulations. In these simulations, as expected, the actual probability of false alarm approached the desired probability of false alarm as the sample size N increased.

From these results, it can be concluded that the asymptotic H_0 distribution is accurate, and it can be used to obtain detection thresholds for detectors with moderately to large sample sizes.

B. Asymptotic Performance Analysis

In all previous research, the performance of a multiple-channel detector based on the GC estimate was evaluated from simulations. Using the asymptotic distributions derived in the previous section, it is possible to develop a theoretical performance analysis.

Table I compares detection thresholds obtained from the exact H_0 distribution that was developed in [1] and thresholds obtained from the asymptotic H_0 distribution for a three-channel GC-based detector corresponding to certain probabilities of false alarm and different sample sizes N. Even, for small sample sizes, the thresholds obtained from the asymptotic distribution give a very good approximation of the exact thresholds. In the table, the largest absolute difference between an asymptotic and an exact threshold is of the order 10^{-6}.

ROC curves for a multiple-channel detector based on the GC estimate can be obtained by noting that the asymptotic distribution of the GC estimate depends on the GC coefficient and the structure of the covariance matrix Σ. Substituting (6), the asymptotic distribution becomes a function of the SNRs of the channels.

Figs. 1 and 2 show ROC curves derived from the asymptotic distribution compared with the result of Monte Carlo simulations for different equichannel SNRs and different vector sizes. In Fig. 1, which considers the case of $M = 3$, the exact H_0 distribution is used, whereas in Fig. 2, which presents ROC curves for $M = 4$ and $M = 5$ channels, the detection thresholds are computed from the asymptotic H_0 distribution. From Fig. 1, it can be observed that even for small vector lengths, the theoretical and the empirically determined curves agree well, as long as the SNR is not too small.

While the GC estimate is asymptotically unbiased, simulations have shown that a better approximation of the distribution for finite values of N is achieved if the asymptotic distribution is corrected by addition of the estimate bias. While a closed-form expression for the bias of the GC estimate given a nonzero value of γ^2 is not known, many problems of practical interest involve SNRs sufficiently small that it is reasonable to approximate the bias by assuming $\gamma^2 = 0$, which is given in (4). This bias correction factor was used in Figs. 1 and 2.

C. *Number of Channels Versus Performance*

One consideration in applying a multiple-channel detection statistic to a problem is the choice of the number of channels. A tradeoff exists between computational expense and a higher probability of detection as the number of channels increases. Table III shows equichannel SNRs to achieve certain probabilities of detection for GC-based detectors with different number of channels M. For fixed sample size N, P_{fa}, and P_d, it can be observed that the required SNR reduces as the number of channels increases.

This increase in performance is also documented in Table IV. An M-channel GC-based detector operates at a certain SNR and achieves a certain probability of detection. The table shows the required SNR of an additional channel that the resulting $(M+1)$-channel detector achieves a similar performance as the M-channel detector. The required SNRs of the additional channel are, in general, about 3–5 dB lower than the SNRs of the other M channels. Therefore, the probability of detection of a detector can be increased as long as the SNR of any additional channel is, at most, 3 dB lower than the minimum SNR of the other channels.

D. *Comparison with Multiple Coherence*

Another important nonparametric detection statistic is the multiple coherence (MC) estimate. It has been used in several applications [8], [16], [17]. This section presents a comparison of the detection performance of a GC-based detector to a detector based on the MC-estimate. In [1], results from simulations were used for a comparison of three-channel detectors.

For this comparison, it is assumed that a common white Gaussian signal is present in independent additive white Gaussian noise. Under these conditions, the MC estimate of channel i is defined [9] as

$$\hat{\gamma}^2_{\mathrm{MC}}(i\colon 1, \cdots, i-1, i+1, \cdots, M) \triangleq 1 - \frac{1}{A_{ii}A^{ii}}$$

TABLE III
EQUICHANNEL SNR NECESSARY TO ACHIEVE CERTAIN PROBABILITIES OF DETECTION FOR DESIRED PROBABILITIES OF FALSE ALARM, $M = 4$, 5, AND 10 CHANNELS, AND SAMPLE SIZES $N = 256$, 512, AND 1024

M	P_{fa}	*N* = 256			*N* = 512			*N* = 1024		
		0.5	0.7	0.9	0.5	0.7	0.9	0.5	0.7	0.9
4	10^{-1}	-13.1	-12.1	-10.7	-14.6	-13.7	-12.3	-16.2	-15.3	-14.0
	10^{-2}	-11.2	-10.5	-9.5	-12.9	-12.2	-11.2	-14.4	-13.8	-12.9
	10^{-3}	-10.3	-9.7	-8.8	-11.9	-11.4	-10.5	-13.6	-13.0	-12.2
	10^{-4}	-9.6	-9.1	-8.3	-11.3	-10.8	-10.1	-13.0	-12.5	-11.8
	10^{-5}	-9.1	-8.6	-7.9	-10.8	-10.4	-9.7	-12.5	-12.0	-11.4
5	10^{-1}	-13.6	-12.7	-11.5	-15.2	-14.4	-13.1	-16.8	-15.9	-14.8
	10^{-2}	-11.9	-11.2	-10.3	-13.5	-12.9	-12.0	-15.1	-14.5	-13.6
	10^{-3}	-11.0	-10.4	-9.6	-12.6	-12.1	-11.3	-14.3	-13.8	-13.0
	10^{-4}	-10.4	-9.9	-9.1	-12.0	-11.6	-10.9	-13.7	-13.2	-12.6
	10^{-5}	-9.9	-9.4	-8.8	-11.6	-11.2	-10.5	-13.2	-12.8	-12.2
10	10^{-1}	-15.5	-14.8	-13.8	-17.0	-16.3	-15.4	-18.5	-17.9	-17.0
	10^{-2}	-13.9	-13.4	-12.7	-15.4	-14.9	-14.2	-16.9	-16.5	-15.9
	10^{-3}	-13.2	-12.7	-12.0	-14.6	-14.2	-13.6	-16.2	-15.8	-15.2
	10^{-4}	-12.6	-12.2	-11.6	-14.1	-13.7	-13.2	-15.7	-15.3	-14.8
	10^{-5}	-12.2	-11.8	-11.2	-13.7	-13.4	-12.8	-15.3	-15.0	-14.5

TABLE IV
REQUIRED SNR OF AN ADDITIONAL CHANNEL THAT AN $(M+1)$-CHANNEL GC DETECTOR OBTAINS EQUAL PROBABILITY OF DETECTION AS A M-CHANNEL GC-BASED DETECTOR OPERATING AT A CERTAIN EQUICHANNEL SNR $= -6$ dB, -9 dB, -12 dB FOR $M = 4$, 5, AND 10 CHANNELS, AND $N = 256$, 512, AND 1024

M	P_{fa}	*N* = 256			*N* = 512			*N* = 1024		
		-6dB	-9dB	-12dB	-6dB	-9dB	-12dB	-6dB	-9dB	-12dB
3	10^{-1}	-11.5	-14.4	-15.6	-12.0	-15.7	-17.1	-12.3	-16.7	-18.7
	10^{-2}	-10.9	-13.6	-15.2	-11.6	-14.8	-16.2	-12.1	-15.9	-17.6
	10^{-3}	-10.7	-13.4	-15.4	-11.4	-14.4	-16.1	-11.9	-15.5	-17.2
	10^{-4}	-10.6	-13.4	-15.6	-11.2	-14.3	-16.2	-11.8	-15.3	-17.1
	10^{-5}	-10.5	-13.4	-15.9	-11.1	-14.2	-16.3	-11.7	-15.2	-17.1
4	10^{-1}	-10.5	-14.2	-15.9	-10.8	-15.1	-17.3	-11.0	-15.7	-18.8
	10^{-2}	-10.1	-13.4	-15.3	-10.5	-14.3	-16.3	-10.8	-15.2	-17.6
	10^{-3}	-9.9	-13.1	-15.2	-10.4	-14.0	-16.0	-10.5	-14.9	-17.2
	10^{-4}	-9.8	-13.0	-15.3	-10.2	-13.7	-15.9	-10.4	-14.7	-17.0
	10^{-5}	-9.7	-12.9	-15.4	-10.1	-13.6	-15.9	-10.2	-14.6	-16.8
5	10^{-1}	-9.7	-13.9	-16.2	-9.9	-14.5	-17.4	-9.8	-14.9	-18.6
	10^{-2}	-9.5	-13.2	-15.4	-9.7	-13.9	-16.4	-9.6	-14.5	-17.6
	10^{-3}	-9.3	-12.9	-15.3	-9.6	-13.6	-16.0	-9.6	-14.3	-17.1
	10^{-4}	-9.2	-12.8	-15.2	-9.5	-13.4	-15.8	-9.5	-14.1	-16.9
	10^{-5}	-9.2	-12.7	-15.3	-9.5	-13.3	-15.8	-9.5	-14.0	-16.7

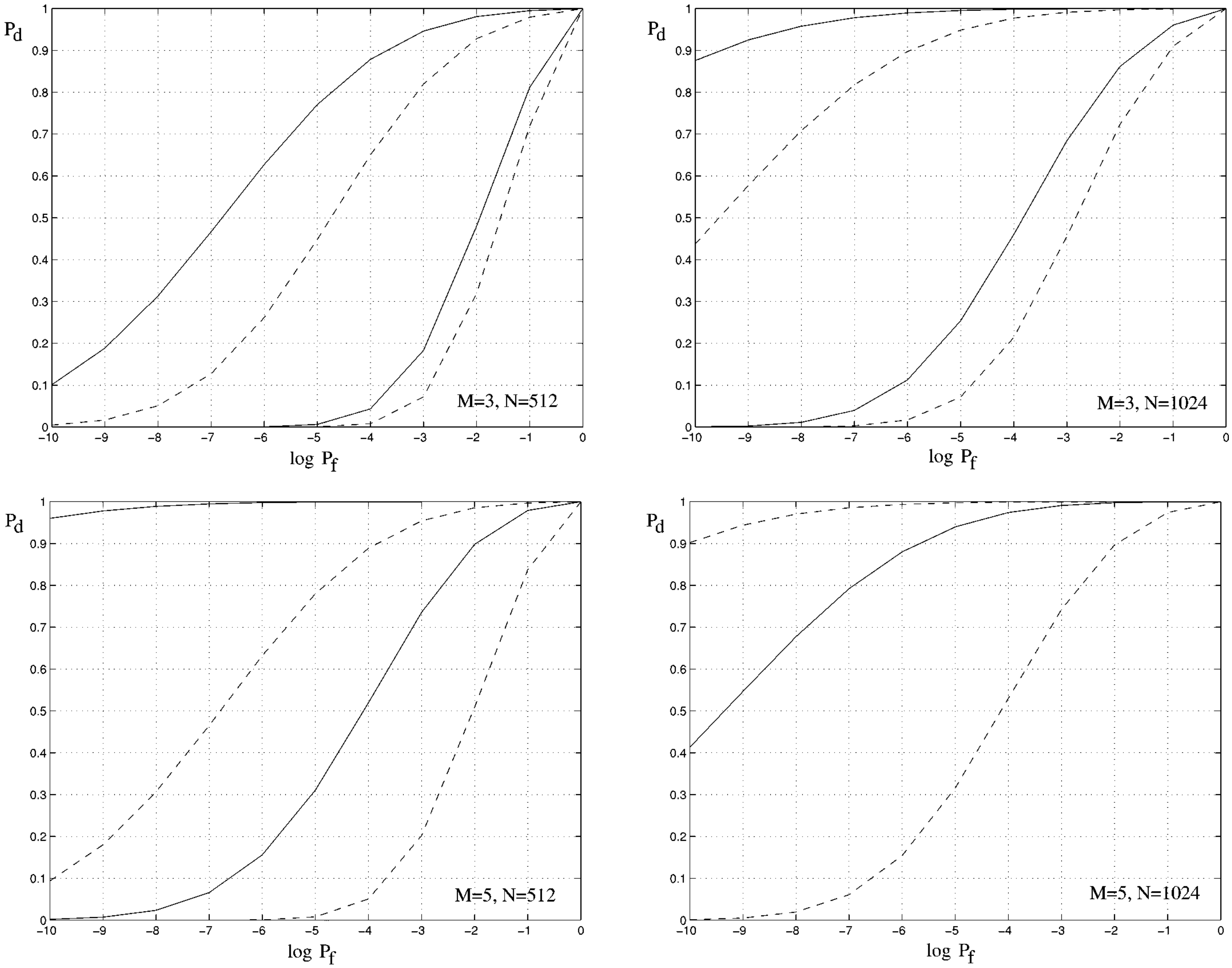

Fig. 3. Comparison of the performance of a GC-based detector (solid lines) to a MC-based detector (dashed lines) with equichannel SNR. The upper curves are for SNR $= -9$ dB and lower curves are for SNR $= -12$ dB. For $M = 5,\ N = 1024$ the GC-based detector achieves $P_d = 1$ for all $P_f > 10^{-10}$.

where A_{ii} is the ith diagonal element of A, as defined in (2), and A^{ii} is the ith diagonal element of the inverse of A. The MC estimate is not invariant to the reordering of the channels, but it relies on a "reference channel." Therefore, for all comparisons presented in this section, it is assumed that the M channels have equal SNR. A detector based on this estimate was introduced in [8], and thresholds for the detector are computed from the exact H_0 distribution as given in that reference. Similar to the GC-based detector, the ROC curves are obtained by expressing the MC coefficient in terms of the SNRs of the channels and an asymptotic non-null distribution obtained from a theorem given in [14].

In [11] and [18], it is shown that the MC estimate is optimal to determine a linear time-invariant relationship between the M channels. It is noted that the optimality of the MC estimate does not apply to the comparison in this section because the relationship to be detected among the M channels is not linear.

In Fig. 3, ROC curves for a GC-based detector are compared with ROC curves for a MC-based detector. The results for the GC-based detector are obtained from the asymptotic performance analysis developed previously. The graphs show comparisons for $M = 3, 5$, $N = 512, 1024$ and SNR $= -9$ dB and -12 dB. It can be observed that the GC-based detector outperforms the MC-based detector in all cases considered. The margin becomes bigger as the number of channels increases.

V. Discussion and Conclusions

The asymptotic distributions for the GC estimate derived in this paper allow a much more complete understanding of the characteristics of GC-based detectors than was previously possible. Although it remains to develop precise bounds on the error resulting when the asymptotic expressions are used to approximate the actual distributions arising from GC estimation with finite sample sequences, empirical evidence presented here suggests that the error is sufficiently small to make the asymptotic expressions very useful in predicting detector performance—even for relatively small sample sequences.

Use of the asymptotic H_0 distribution greatly simplifies calculation of detection thresholds when more than three channels are involved. The availability of the asymptotic distribution for H_1 scenarios with additive Gaussian signal provides a foundation for a much more complete analysis of the performance of GC-based detectors in this important setting than was previously possible. In addition to removing the need for extensive computer simulation to predict detector performance, this analytical capability allows direct performance comparison between GC- and MC-based detectors—some of which has been undertaken in this paper. It also makes questions regarding the marginal value of additional data possible to address. Results presented in this paper show that adding another channel to a multiple-channel detection statistic can increase the performance of the detector (obviously at the expense of computational complexity). The authors expect to look more closely at such questions in future research with the goal of quantifying tradeoffs between number of channels, SNR, integration time, and processing bandwidth.

It is important to stress that the asymptotic non-null distribution of the GC-estimate derived in this paper is for the special case of a white Gaussian signal in additive white Gaussian noise. Further research is necessary to obtain results under other H_0 and H_1 assumptions, including colored and non-Gaussian signals and noise. In the meantime, asymptotic results (e.g., [19]) suggest that the hypotheses used in this paper might be useful approximations when dealing with broadband signals and noise having non-Gaussian distributions that have been passed through narrowband filters. Moreover, it is still possible to use GC-based CFAR tests to detect deviations from H_0 arising from means other than addition of a common white Gaussian signal component on each channel.

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