On the Preservation of Bandlimitedness Under Non-affine Time Warping

Shahrnaz Azizi, Douglas Cochran, and John N. McDonald
Arizona State University
Tempe, AZ 85287 USA

Abstract

Time-warped functions, and time-warped bandlimited functions in particular, have received attention in connection with nonuniform sampling. A conjecture to the effect that bandlimitedness is preserved under continuous and bijective time warping only if the warping function is affine has recently been shown to be false by an example constructed by Y. Meyer. This opens theoretical questions about conditions under which bandlimitedness is preserved by time warping that have significant practical implications in sampling and reconstruction of time-warped bandlimited functions. This paper examines some of these questions and presents some new results on the preservation of bandlimitedness by non-affine time warping.

1. Introduction

A bandlimited function f in $L^2(\mathbb{R})$ is one of the form

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \hat{f}(\omega)e^{i\omega t}\, d\omega$$

where $f \in L^2(\mathbb{R})$ and $0 < \Omega < \infty$. A time-warped bandlimited function $h : \mathbb{R} \to \mathbb{C}$ is defined by $h = f \circ \gamma$ where f is bandlimited and $\gamma : \mathbb{R} \to \mathbb{R}$. In [1], J.J. Clark *et al.* observed that, if γ is invertible, the time-warped bandlimited function h can be reconstructed from samples $\{h(t_n) = h\left(\gamma^{-1}(nT)\right) : n \in \mathbb{Z}\}$ where $T = \pi/\Omega$ is the Nyquist interval for f. This follows immediately from the Shannon sampling theorem: $h(t_n) = f\left(\gamma\left(\gamma^{-1}(nT)\right)\right) = f(nT)$ are Nyquist samples of f so that

$$f(t) = \sum_{n\in\mathbb{Z}} h(t_n)\phi_n(t)$$

with $\phi_n(t) = \operatorname{sinc}\left[T^{-1}(t - nT)\right]$, where

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Hence

$$h(t) = f\left(\gamma(t)\right) = \sum_{n\in\mathbb{Z}} h(t_n)[\phi_n \circ \gamma](t) \qquad (1)$$

If γ is affine (i.e., $\gamma(t) = at + b$ with $a \neq 0$), h is bandlimited and the samples $h(t_n)$ are uniformly spaced. It was noted in [2] that, if γ is not affine, h will generally not be bandlimited and the sampling times t_n will generally not be uniformly spaced. Thus, Clark's observation provides a means for reconstructing certain non-bandlimited functions from uniformly spaced samples.

Also in [2], it was conjectured that if $f \circ \gamma$ is bandlimited with f bandlimited and γ continuous and bijective, then γ must be affine. One key ramification of this conjecture is representational uniqueness. Suppose $h = f_1 \circ \gamma_1 = f_2 \circ \gamma_2$ with f_1 and f_2 bandlimited and γ_1 and γ_2 continuous and bijective. Then $f_1 = f_2 \circ (\gamma_2 \circ \gamma_1^{-1})$ is bandlimited and hence $\gamma_2 \circ \gamma_1^{-1}$ is affine; i.e., the representation of h as a time-warped bandlimited function is unique up to an affine "factor" in the warping function. This provides hope for developing an implicit sampling procedure for time-warped bandlimited functions $h = f \circ \gamma$ in which f and γ (and thus the sampling times t_n and interpolation functions $\phi_n \circ \gamma$) are deduced from the values of h. The veracity of the conjecture would also imply some uniqueness results connecting bandlimited functions and their values at the zero crossing of their derivatives, without explicit reference to the zero crossing times.

The conjecture was shown to be false with a counterexample constructed by Y. Meyer in 1994, opening questions regarding conditions under which a bandlimited function f remains bandlimited after non-affine time warping γ. This paper provides some results indicating that the circumstances under which this can occur are rather specific and require specific combinations of f and γ (i.e., there is no non-affine γ that preserves bandlimitedness for all f).

Before proceeding to describe Meyer's example (which is previously unpublished) and presenting the main results of this paper, a few comments related to the topic of time-warped signals are summarized below.

- Clark's observation applies to any sampling theorem of the form

  $$f(t) = \sum_{n\in\mathbb{Z}} f(\tau_n)\phi_n(t)$$

  to provide a reconstruction formula of the form (1) with $t_n = \gamma^{-1}(\tau_n)$ for time warped functions.

- It has been noted [3] that Clark's sampling result for time-warped bandlimited functions is implied by Kramer's theorem [4]. More recently, sampling formulae of the form (1) for time-warped Kramer spaces have been observed to follow both from Kramer's theorem and from the reproducing kernel Hilbert space structure of such spaces [5].

- Time-warped signals have received much recent attention in the signal processing literature in connection with numerous applications (see for example,[6, 7, 8, 9, 10]).

2. Meyer's Example

The following example, taken from [11], constructs two bandlimited functions f_1 and f_2 such that $f_1 = f_2 \circ \gamma$ where γ is continuous and bijective, but not affine.

Example 1 (Meyer) *1* Start with

$$y_1(t) = \frac{\sin^8(t)}{t^7}$$

which is an odd function on the line. Define

$$f_1(t) = \int_t^\infty \frac{\sin^8(x)}{x^7}\,dx$$

Then f_1 is even, $f_1(0) > 0$,

$$f_1'(t) = -\frac{\sin^8(t)}{t^7},$$

f_1 is strictly decreasing on $[0,\infty)$, and f_1 is strictly increasing on $(-\infty,0]$. Finally, multiply f_1 by $c > 0$ in order to achieve $f_1(0) = 1$.

By starting with

$$y_2(t) = \frac{\sin^{10}(t)}{t^9}$$

and doing the same, f_2 is obtained such that $f_2(0) = f_1(0) = 1$. The graph of f_2 is similar to that of f_1. Both f_1 and f_2 are bandlimited functions. Indeed, $\hat{y}_1(\omega) = -i\frac{d}{d\omega}\varphi(\omega)$ where φ is a *basic spline* supported by $[-8,8]$ and $\hat{f}_1(\omega) = \varphi'(\omega)/\omega$ also supported by $[-8,8]$.

Finally $\gamma(t)$ is defined on $[0,\infty)$ by $\gamma(t) = u \Leftrightarrow f_1(t) = f_2(u)$, which is meaningful since f_1 maps $[0,\infty)$ onto $(0,1]$ and f_2 is 1-1 between $[0,\infty)$ and $(0,1]$. Now γ is clearly a homeomorphism, $\gamma(t)$ is extended to $\mathbb{R}$ by $\gamma(-t) = -\gamma(t)$ and the construction of the warping function is complete.

In order to get a real analytic γ, one can start with $y_1(t) + y_1(\sqrt{2}t)$ instead of $y_1(t)$.

3. Warping Functions that Preserve Bandlimitedness

Faced with the failure of the general conjecture, it is desirable to determine conditions under which preservation of bandlimitedness is only achieved by affine warping functions. The main result of this paper is the following:

Corollary 1 *If γ is continuous and invertible and $f \circ \gamma$ is bandlimited for every bandlimited f, then γ is affine.*

This establishes that preservation of bandlimitedness is not a characteristic of the warping function except in the affine case. The warping function γ in Meyer's example, for instance, preserves bandlimitedness when it is used to warp f_1. The Corollary implies that there exist other bandlimited functions for which bandlimitedness is not preserved when they are composed with this same γ. The corollary follows from a result specific to time-warped sinc functions:

Theorem 1 *If $\operatorname{sinc}(\gamma(\cdot))$ and $\operatorname{sinc}(\gamma(\cdot) - n)$, $n \neq 0 \in \mathbb{Z}$ are bandlimited and $\gamma : \mathbb{R} \to \mathbb{R}$ is continuous and invertible, then γ is affine.*

The proof begins by extending γ to the complex plane. Denoting $F_0(t) = \operatorname{sinc}(\gamma(t))$ and $F_n(t) = \operatorname{sinc}(\gamma(t) - n)$, a straightforward calculation yields

$$F_n(t) = (-1)^n F_0(t)\frac{\gamma(t)}{\gamma(t) - n}$$

Thus,

$$\gamma(t) = \frac{nF_n(t)}{F_n(t) - (-1)^n F_0(t)}$$

Since F_0 and F_n are bandlimited by assumption, they extend to entire functions and γ is seen to agree with a meromorphic function Γ on the real line. The function sinc is entire and hence is singular at infinity. Thus, if ζ were a pole of Γ, F_0 would also be singular at ζ. Because F_0 is entire, this cannot happen for finite ζ and Γ is thus entire.

Having established that Γ is entire, it is now possible to apply classic growth results for compositions of entire functions to establish that Γ is affine.

A theorem of G. Pólya [12] establishes that if a composition $\Lambda \circ \Gamma$ of entire functions is of finite order [13] and the order of Λ is non-zero and finite, then Γ is a polynomial. The order of the function sinc is exactly one, so Γ is a polynomial.

It remains to establish that the degree of Γ is one. For an arbitrary entire function G, denote $M_G(r) = \sup\{|G(z)| : |z| \leq r\}$ for $r \geq 0$. A second result of Pólya, also from [12], shows that if $G = \Lambda \circ \Gamma$ with Λ and Γ entire and $\Gamma(0) = 0$, then there exists a constant $c \in (0,1)$ such that $M_G(r) \geq M_\Lambda(cM_\Gamma(r/2))$ for all $r > 0$. In the case at hand, Λ is a sinc function and Γ is a polynomial of degree $n \geq 1$. Since Γ is bijective on $\mathbb{R}$, n must be odd and Γ has exactly one real root, which can clearly be assumed to be zero without loss of generality. If $n \neq 1$, then $n \geq 3$ and $|\Gamma(t)| > t^2$ for t sufficiently large. Therefore $M_\Gamma(r) > r^2$ for all sufficiently large r. The maximum modulus principle [14] implies that M_Λ is a monotone increasing function. Thus there is some $c > 0$ such that $M_G(r) \geq M_\Lambda(cM_\Gamma(r/2)) > M_\Lambda(cr^2/4)$. Hence the order of G is

$$\begin{aligned}\rho_G &= \limsup_{r\to\infty} \frac{\log\log M_G(r)}{\log r} \\ &> \limsup_{r\to\infty} \frac{\log\log M_\Lambda(cr^2/4)}{\log r} \\ &> \limsup_{r\to\infty} \frac{\log\log M_\Lambda(r)}{\log r} = \rho_\Lambda\end{aligned}$$

Since $\rho_\Lambda = 1$ and $\rho_G \leq 1$, this cannot hold and the degree of Γ is one, as desired. ∎

Corollary 1 follows immediately: the same non-affine γ cannot preserve the bandlimitedness of both F_0 and F_n, let alone all bandlimited functions.

4. Series Representation

Because every bandlimited function f has a series representation in shifted sinc functions, every time-warped bandlimited function $g = f \circ \gamma$ has such a series representation in shifted time-warped sinc functions. If γ is not affine, Theorem 1 implies that g has a series expansion in which at most one term is bandlimited. In this section, after giving a modified form of Theorem 1, it will be shown that in fact g has a series expansion in which no term is bandlimited.

Theorem 2 *Let $\gamma : \mathbb{R} \to \mathbb{R}$ be continuous and invertible and, for $\eta \in \mathbb{R}$, denote*

$$F_\eta(t) = \operatorname{sinc}\left(\frac{\gamma(t) - \eta T}{T}\right)$$

If either of the following holds, then γ is affine.

(I) The three functions F_0, $F_{1/2}$, and F_α with α not equal to zero or $1/2$ are bandlimited.

(II) The two functions F_α and F_β with $\alpha \neq \beta$ and $\alpha - \beta \in \mathbb{Z}$ are bandlimited.

As with Theorem 1, the proof requires extension of γ to the complex plane. Considering F_η, $\eta \in \{0, \frac{1}{2}, \alpha, \beta\}$, and using $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$ yields

$$F_\eta(t) = F_0(t)\frac{\gamma(t)\cos(\pi\eta)}{\gamma(t) - \eta T} + F_{\frac{1}{2}}(t)\frac{\left(\gamma(t) - \frac{1}{2}T\right)\sin(\pi\eta)}{\gamma(t) - \eta T}$$

Thus,

$$\gamma(t) = \frac{\alpha T\sin(\pi\beta)F_\alpha(t) - \beta T\sin(\pi\alpha)F_\beta(t)}{F_\alpha(t)\sin(\pi\beta) - F_\beta(t)\sin(\pi\alpha) - F_0(t)\sin\pi(\beta-\alpha)}$$

Note that the denominator and numerator in this expression cannot simultaneously vanish except possibly at isolated points. Hence the rest of the proof follows from Theorem 1 and that the function $\operatorname{sinc}(T^{-1}(\cdot))$ is entire with order one. ∎

Theorem 2 implies that if any combination of two distinct shifted sinc functions, as well as the unshifted sinc function warp to bandlimited functions, then the warping function has to be affine. Also, if two distinct shifted sinc functions whose shifts are separated by an integer warp to bandlimited functions, then γ has to be affine. When this result is applied to standard series representations for a time-warped bandlimited function, it shows that $g = f \circ \gamma$ has a series expansion in which no term is bandlimited.

For f bandlimited,

$$\begin{aligned} f(t) &= \sum f(nT)(F_0 \circ \gamma^{-1})(t - nT) \\ &= \sum f(nT + T/2)(F_{1/2} \circ \gamma^{-1})(t - nT) \\ &= \sum f(nT + \alpha T)(F_\alpha \circ \gamma^{-1})(t - nT)\end{aligned}$$

Hence every time-warped bandlimited function $g = f \circ \gamma$ has series representations

$$\begin{aligned} g(t) &= \sum g(\gamma^{-1}(nT))F_0(t - nT) \\ &= \sum g(\gamma^{-1}(nT + T/2))F_{1/2}(t - nT) \\ &= \sum g(\gamma^{-1}(nT + \alpha T)F_\alpha(t - nT)\end{aligned}$$

At least one of the above series has no bandlimited term, otherwise γ would be affine by Theorem 2.

The point of arguing that a time-warped bandlimited function has such a series representation in non-bandlimited functions is to illustrate that preservation of bandlimitedness under non-affine warping is a delicate property; it relies on cancelation of the terms of the series outside the band and hence can be destroyed by changing a single coefficient (sample value) in the series expansion of f. A topological consequence of this is that, if f and $f \circ \gamma$ are bandlimited with γ non-affine, then every neighborhood of f contains bandlimited functions whose bandlimitedness is not preserved under warping by γ.

5. Conclusion

This paper has examined some basic questions about conditions under which bandlimitedness is preserved by time warping. The main results show that, although it is possible for bandlimitedness to be preserved under non-affine warping, that it is not possible for a non-affine warping function to preserve bandlimitedness in general.

Ongoing research by the authors shows that the collection of non-affine warping functions that can preserve bandlimitedness for even a single bandlimited f is small in certain rigorous senses. It is hoped that further results in this direction will lead to recovery of some of the promise of the original conjecture.

References

[1] J.J. Clark, M.R. Palmer, and P.D. Lawrence, "A transformation method for the reconstruction of functions from nonuniformly spaced samples," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-33(10), pp. 1151–1165, October 1985.

[2] D. Cochran and J.J. Clark, "On the sampling and reconstruction of time-warped bandlimited signals," *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, April 1990.

[3] Y.Y. Zeevi and E. Shlomot, "Nonuniform sampling and antialiasing in image representation," *IEEE Transactions on Signal Processing*, vol. SP-41(3), pp. 1223–1236, March 1993.

[4] H.P. Kramer, "A generalized sampling theorem," *Journal of Mathematics and Physics*, vol. 38, pp. 68–72, 1959.

[5] S. Azizi and D. Cochran, "Reproducing kernel structure and sampling on time-warped Kramer spaces," *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, March, 1999.

[6] R.G. Baraniuk and D.L. Jones, "Warped wavelet bases: Unitary equivalence and signal processing", *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing*, 1993, vol. III, pp. 320–323.

[7] R.G. Baraniuk and D.L. Jones, "Shear madness: New orthonormal bases and frames using Chirp Functions", *IEEE Transactions on Signal Processing*, vol. 41, No. 12, pp. 3543–3548 Dec. 1993.

[8] R.G. Baraniuk, "Warping time-frequency and time-scale representations to match signals," *Proceedings of the SPIE International Society for Optical Engineering*, vol. 2488, pp. 34-45, 1995.

[9] G.F. Boudreaux-Bartels, "On the use of operators versus warpings versus axiomatic definitions of new time-frequency (operator) representations," *Proceedings of 28th Asilomar Conference on Signals, Systems, and Computers*, Monterey, CA, pp. 36-40, 1994.

[10] G. Evangelista and S. Cavaliere, "Discrete frequency warped wavelets: Theory and applications," *IEEE Transactions on Signal Processing*, vol. 46, no. 4, pp. 874-885, April 1998.

[11] Y. Meyer, Private Correspondence, 1994.

[12] G. Pólya, "On an integral function of an integral function," *Journal of the London Mathematical Society*, vol. 1, pp. 12–15, 1926.

[13] R.P. Boas, Jr., *Entire Functions*, Academic Press, New York, 1954.

[14] W. Rudin *Real and Complex Analysis*, McGraw-Hill series in higher mathematics. McGraw-Hill, 1974.