Hybrid Wavelet Packets
A Top Down Approach

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Abstract

Hybrid wavelet packets have recently been introduced as a method which addresses the observation that the performance of a transform coding scheme may be improved by allowing the choice of QMF to be included within the analysis. The choice of appropriate QMF is not only signal dependent, but scale dependent within a given signal as well. A top-down scheme for determining a well suited hybrid wavelet packet basis is presented. This method greatly reduces the computational complexity of previous schemes allowing deeper search trees to be computed and demonstrates an associated improvement in performance.

1. Introduction

Hybrid wavelet packets are a generalization of wavelet packets which include the choice of wavelet function within the analysis [1]. This is motivated by the observation that the short time Fourier transform "depends critically on the choice of the window [function]" [4]. This same critical dependence holds between the choice of wavelet and the resulting wavelet decomposition; this dependence is seldom explicitly addressed in the development of compression algorithms. "A very common pitfall when using any kind of transform is to forget the presence of the analyzing function in the transformed field." [5]

Hybrid wavelet packets are conceptually a direct extension of wavelet packets, however computationally this direct extension quickly becomes inviable. From their inception, it has been obvious that direct implementation is only possible for small cases.

This paper introduces a top-down scheme for optimizing hybrid wavelet packets. This approach uses local optimization to determine the hybrid wavelet packet basis which, although sub-optimal, provides a tremendous reduction in computation making the method much more applicable.

In Section 2 wavelet packets will be briefly reviewed. Hybrid wavelet packets will then be introduced by generalizing wavelet packets to include multiple QMFs. The multiplicity of bases that result from hybrid wavelet packet analysis is large enough that exhaustive tree searching strategies, such as the best basis algorithm [3], quickly become inviable. The top-down algorithm will be described and computational issues will be discussed.

In Section 3 experimental results will be presented. The results indicate that choice of QMF does effect the compressibility of the decomposition and the hybrid wavelet packets do provide bases that have a higher degree of compressibility than standard best basis decompositions for a class of signals. This difference becomes more prominent with increased maximum allowable tree depth.

2. From Wavelet Packets to Hybrid Wavelet Packets

2.1. Wavelet Packets

The discrete wavelet transform (DWT) can be characterized as a recursive application of the highpass and lowpass filters that form a QMF pair. The calculation of the DWT begins by filtering a signal by the highpass and lowpass filters and then downsampling the output. The computation proceeds by applying the QMF pair to the output of the lowpass filter. The recursion, then, is simply just a repeated application of the QMF pair to the lowpass filtered output of the previous level. Wavelet packets are generated by only slightly changing this operation.

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To calculate a wavelet packet decomposition, the procedure begins as before, with the application of the QMF to the data followed by downsampling. However, now the computation proceeds by applying the QMF to not only the lowpass output but to the highpass output as well. The recursion is simply to filter and downsample all output of the previous level. The calculation of wavelet packets is often schematically characterized by the formation of a binary tree with each branch representing the highpass and lowpass filtered output of a root node.

Definition 0.1 *A **tableau** is a wavelet packet tree. It is a structure for organizing the output of the recursive applications of a single QMF pair in a wavelet packet expansion.*

2.2. Hybrid Wavelet Packets

The generation of a standard wavelet packet library of bases relies on the initial, possibly arbitrary, choice of a QMF, often chosen without regard to the effect on performance. Although an experimental researcher will typically have some intuition for which wavelet to choose for a given problem, it is desirable to develop methods to reliably select bases which are well suited for representation of particular signals.

Accepting that the choice of QMF inherently affects the performance of a compression scheme, a simple solution is to perform a best basis analysis for m different QMFs and then choose the "best of the best." This offers the possibility of improved compression, but this simple approach is merely a superficial use of multiple QMFs.

The central concept of this research is that the choice of appropriate QMF is not only signal dependent but scale dependent within a given signal as well. That is, given compression as the ultimate goal, the choice of QMF which yields the best performance may change at different levels within a wavelet packet analysis.

The hybrid wavelet packet algorithm consists of the recursive application of a collection of QMF pairs. The process begins with application of m filters, followed by downsampling. The resultant outputs are then filtered by all m QMFs. The recursion is similar to the wavelet packet calculation in that all outputs are filtered, the difference is that at each level we apply multiple filters. Analogously, hybrid wavelet packets generate a $2m$-ary tree, with each root node branching into m pairs of highpass and lowpass outputs.

The motivation for the use of hybrid wavelet packets is that for a given signal and a *collection of QMF pairs*, it is possible that one of the resulting hybrid bases contain basis functions that are more highly correlated with the signal than would be possible with a standard wavelet packet basis. This degree of correlation translates into a degree of compressibility. Hence, we can improve performance by optimizing the choice of QMF for each level of the wavelet packet analysis.

An example of a signal with such hybrid content is a signal that is not self-similar, i.e., one in which the small scale properties are different than the large scale properties, as in the pulse modulated narrow-band signal depicted in Figure 1.

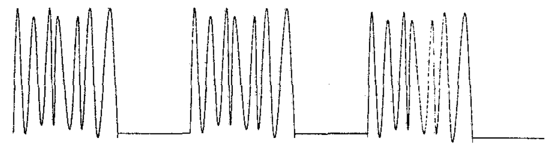

Figure 1. A signal with large scale properties similar to a Haar wavelet but with small scale properties which resemble a higher order Daubechies wavelet.

Definition 0.2 *A **hybrid tableau** is a hybrid wavelet packet tree in which a pair of QMFs has been chosen at each level, thus resulting in a binary wavelet packet tree with* mixed QMFs.

One of the arguments for the use of wavelet packets is that a large collection of bases is generated from which we can choose a basis which yields an optimal representation. The growth in the size of the collection of basis from level L to level $L+1$ is given by the recursive relation [3]

$$A_{L+1} = A_L^2 + 1.$$

A similar result for a hybrid wavelet packet expansion follows by similar calculation.

Theorem 1 *Given m filter banks, the number of bases contained in an $L+1$ level $2m$-ary tree generated by a hybrid wavelet packet analysis is given recursively by*

$$A_{L+1} = m(A_L^2 + 1),$$

where A_L is the number of bases at level L.

One implementation is to simply extend the best basis algorithm [3, 6] to the $2m-$ary tree. This consists of calculating the hybrid coefficients, and the associated costs, to level L. For these experiments the cost function used is the modified Shannon entropy as introduced by Coifman et al. A pruning procedure proceeds by comparing the pairwise sum of costs of the terminal branches at level L, with the cost of their parent nodes at level $L-1$. Each node at level $L-1$ will have m pairs of branches corresponding to the coefficients that result from the application of the m QMF pairs.

If the cost of the root node is less than that of its branch pairs, the branches are pruned. However, if any pair of branches has a cost less than the root, the pair with the lowest cost is chosen as terminal node and is part of the final expansion; all other $(m-1)$ pairs are pruned. This method is then applied to the new terminal nodes at level $L-1$ and traverses up the tree until all nodes are terminal nodes.

Given the exponential growth of the tree size, this exhaustive search algorithm quickly becomes inviable for real time applications.

2.3. A Top-Down Optimized Hybrid Tableau

To reduce the computational cost, a top-down method has been developed. In this method a hybrid tableau is generated by locally optimizing the choice of filter pairs in a top-down manner. The procedure begins by applying each of the filter pairs at the root node and calculating the entropy of the resulting decompositions. The filter pair that yields the lowest entropy is selected as the entry for the root node in the hybrid tableau. The iteration proceeds to the next level of the tableau by applying each filter pair to the output of the optimized node and computing the entropy of the decompositions. For each node the filter pair that yields the minimum nodal entropy is added to the tableau. This local optimization process is continued until the final level L is reached. This procedure yields an L level hybrid tableau to which the best basis pruning procedure is applied.

This method produces only one hybrid tableau requiring only one best basis pruning procedure, whereas the previous methods all required the generation and optimization of a large number of hybrid tableaus. A comparison of the complexity for various methods is shown in Table 1. The number of computations required for the top-down scheme is nearly the same as the simple multiple best basis approach, in which a best basis is computed for each QMF in consideration. The only additional computation is the comparison of entropies in the local optimization step. The experiments that follow indicate that hybridization of the wavelet packets does lead to a basis that has lower entropy than any of the constituent bases for a class of signals.

3. Implementation and Results

Previous investigations of hybrid wavelet packets [1, 2] have illustrated the possibility of improved compression but where overwhelmed by computational complexity, thus demonstrating the need for finding a computationally efficient alternative to the extension of the best basis algorithm.

For one investigation, all possible hybrid tableaus for a three level hybrid wavelet packet tree generated from three

Method	filter applications	pruning
Best basis	2^L	1
Hybrid best basis	$2^{(m+L)}$	$2^{(m+L)}$
Multiple best basis	$m \cdot 2^L$	1
Top-down hybrid best basis	$m \cdot 2^L$ + sorting	1

Table 1. A comparison of the complexity involved in the calculation of a basis for various methods.

QMF pairs were computed (Daub6, Coif2, and Sym6). For each hybrid tableau the best basis was calculated using a modified Shannon entropy as the cost functional [6, 8]. The Shannon entropy was chosen as a cost functional due to its direct relation with the concentration of expansion coefficients and hence the compressibility of the expansion. Thus an expansion with lower entropy is associated with higher compressibility.

In Figure 2 the resulting entropies have been plotted. For reference, a line representing the entropy of the best basis for each of the "standard" tableaus i.e., those generated from only one of each of the QMF pairs, has been included. The hybrid best basis method produces a wide range of entropies, and importantly, several hybrid tableaus yield best bases that have entropy lower than any of best bases generated from only one filter pair.

In [2] a two-step optimization procedure was introduced in which the top nodes of a tree were searched exhaustively; once the optimal basis for the short tree was determined, this upper tree was fixed and an exhaustive search for the next level of branches was made.

The results of this two step process indicate that performance improves with depth of the tree as shown in Figure 3. Thus, a scheme that allows one to generate deeper tree, even if sub-optimal, may improve performance.

The top-down algorithm was implemented and tested on various signals for various tree depths. In Figure 4 a graph of entropy as a function of tree depth is presented. For this signal, the hybrid basis has a lower entropy than any of the marginal cases. Moreover, this difference becomes greater with tree depth.

To help interpret the significance of this reduction in entropy an elementary source coding experiment was performed. Recall that the motivation for the minimization entropy was to provide an orthogonal expansion in which a few coefficients are relatively large and thus the remaining coefficients are relatively small. To compare this energy compaction property with that of the standard wavelet packet expansions, a calculation of energy error as a function of the number of zeroed coefficients was made. The iterative algorithm begins by sorting the coefficients in order

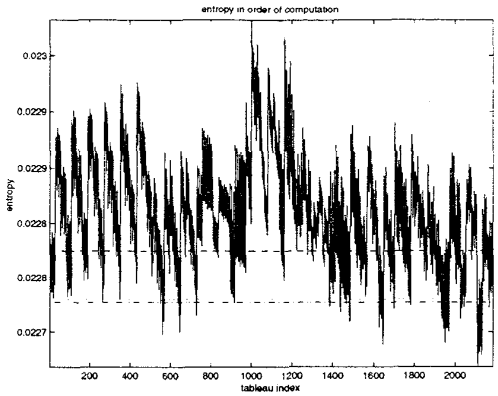

Figure 2. Entropies resulting from the best basis of all possible hybrid tableaus for a three level tree generated from three QMF pairs. The original signal was a quadratic chirp. For reference the entropies resulting from standard best basis calculations using each of the three QMF pairs independently are plotted. There are several hybrid entropy values that are below these marginal reference entropies, demonstrating the existence of hybrid wavelet packets that outperform the standard wavelet packets.

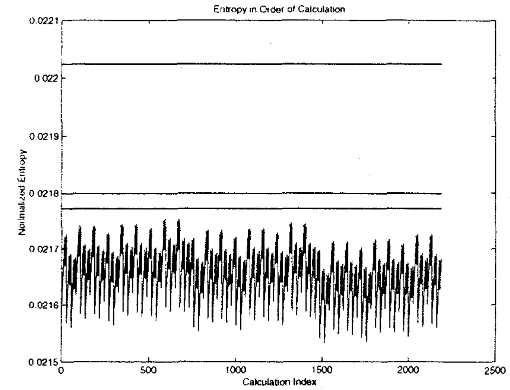

Figure 3. Entropies resulting from the hybrid wavelet packet expansions for a four level tree generated from three QMF pairs. The top three levels of the tree were fixed based on the optimal bases derived from the results shown in Figure 2. The original signal was again a quadratic chirp. The entropies for all of the hybrid cases are now lower than those of the standard wavelet packet expansions.

of magnitude, zeroing all but one coefficient, reconstructing the signal from the non-zero coefficients and measuring the energy of the error. Each subsequent iteration keeps one more non-zero coefficient than its predecessor; this continues until the full expansion is reached.

The results of such an experiment for the wavelet packet trees calculated for a decomposition with a maximum allowable depth of 8 is shown in Figure 5, and for a maximum allowable depth of 12 in Figure 6. The signal was an audio recording of a djembe "slap." For both cases the hybrid wavelet packet basis representation (represented by a solid line) gives lower energy error than the standard wavelet packet basis. It is also apparent that the performance for this signal improves with maximum allowable tree depth.

4. Conclusion

Hybrid wavelet packets generalize wavelet packet analysis to include a choice of QMF at each level within the analysis. This analysis generates a very large collection of bases from which to choose. For compression, we choose a basis such that the inner product of the basis with the signal has minimum entropy.

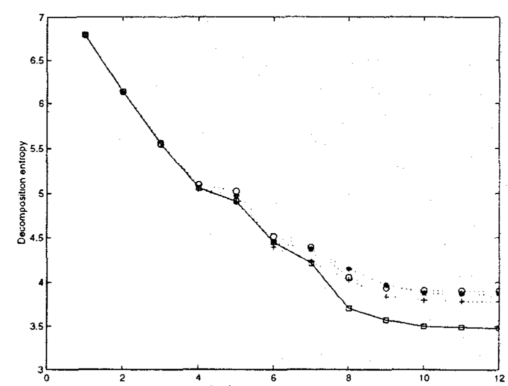

Figure 4. Entropy as a function of maximum allowed tree depth for a hybrid best basis and standard best basis for each of constituent filters. The QMF filters used where Daubechies 6, Coiflet 2, and Symlet 6. The solid line designates the hybrid case and the dashed lines designate the marginal cases. The signal was a 44.1 kHz sample of a Djembe "slap."

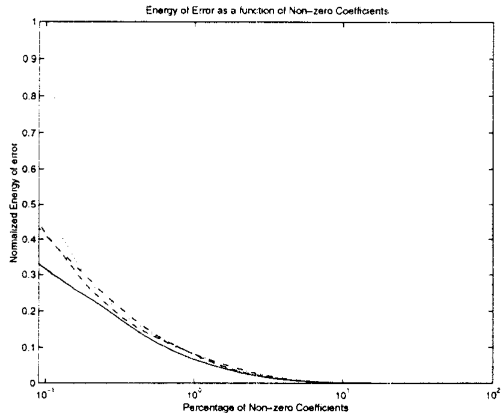

Figure 5. Energy of the approximation error is plotted as a function of the number of non-zero coefficients. The maximum allowable tree depth was 8. The solid line designates the hybrid wavelet packet expansion. For a given number of coefficients the error is smaller for the hybrid wavelet packet case.

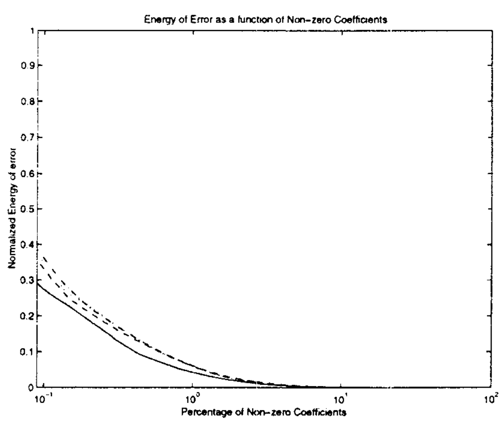

Figure 6. Energy of the approximation error is plotted as a function of the number of non-zero coefficients. The maximum allowable tree depth was 12. The solid line designates the hybrid wavelet packet expansion and show smaller error for a given number of coefficients.

The results show that there are signals for which a hybrid wavelet packet basis provides better compression performance than standard wavelet packet methods. The top-down approach introduced here provides a more computationally efficient, sub-optimal, method to find well suited hybrid wavelet packet bases.

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