Correspondence

An Invariance Property of the Generalized Coherence Estimate

Axel Clausen and Douglas Cochran

***Abstract*— The distribution of the M-channel generalized coherence estimate is shown not to depend on the statistical behavior of the data on one channel, provided that the other $M-1$ channels contain only white Gaussian noise, and all channels are independent. This confirms a recently published conjecture and extends the utility of the GC estimate as a detection statistic to include active as well as passive scenarios.**

I. Introduction

The ability to determine whether a common signal is present on two or more noisy channels is desirable for a variety of applications, particularly in situations where an unknown signal source is to be detected and localized from measurements made at several sensors. In the case of two channels, a commonly used method is to compare the magnitude-squared coherence (MSC) estimate computed using sample sequences from the two channels to a threshold. Because the distribution of the MSC estimate is known under specific H_0 assumptions, detection thresholds corresponding to predetermined probability of false alarm values can be determined [1].

In [2] and [3], the generalized coherence (GC) estimate was introduced as a generalization of the MSC estimate to the case of more than two channels. As with the MSC estimate, the utility of the GC estimate as a detection statistic arises from the fact that its distribution is known under the H_0 hypotheses that all channels are independent and contain stationary white Gaussian noise. This allows detection thresholds corresponding to desired false alarm probabilities to be determined analytically.

In [4], the distribution of the MSC estimate was shown not to depend on the distribution of one channel if the channels are independent and the second channel contains only white Gaussian noise. A geometric argument was used in [5] to show that this result remains valid if the Gaussian assumption is replaced by spherical symmetry. These results led to the conjecture, in [3], that a similar result holds for the GC estimate. A special case of this conjecture was argued in [6].

This correspondence shows that the distribution of the M-channel GC estimate does not depend on the distribution of one channel under the assumptions that all channels are independent, and the other $M-1$ channels contain only stationary white Gaussian noise. One consequence of this result is that the detection thresholds established for use in passive detection can be used directly in a setting in which one of the M channels is an exact replica of a transmitted waveform (assuming, of course, the other H_0 assumptions are met). This suggests the utility of the GC estimate as a detection statistic for "multiple-channel matched filtering" problems in radar, sonar, and seismography. An example of this type of application is given at the end of this correspondence.

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A. Clausen is with the Telecommunications Research Center, Arizona State University, Tempe, AZ 85287-7206 USA.

D. Cochran is with the Department of Electrical Engineering, Arizona State University, Tempe, AZ 85287-5706 USA.

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II. Invariance of Estimate to Distribution of One Channel

For brevity, the notation of [3] is adopted for the remainder of this correspondence. Given M complex N-vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M$, the GC estimate is defined as

$$\hat{\gamma}^2_{M,N}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M) \doteq 1 - \frac{g(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M)}{\|\boldsymbol{x}_1\|^2 \cdots \|\boldsymbol{x}_M\|^2}$$

where

$$g(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M) \doteq \det \begin{bmatrix} \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle & \cdots & \langle \boldsymbol{x}_1, \boldsymbol{x}_M \rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{x}_M, \boldsymbol{x}_1 \rangle & \cdots & \langle \boldsymbol{x}_M, \boldsymbol{x}_M \rangle \end{bmatrix}$$

denotes the determinant of $M \times M$ Gram matrix formed from the sequences $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M$. Properties of Gram matrix determinants are discussed in [7].

Under hypothesis H_0, the following are assumed:

1) The vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_M$ are statistically independent.
2) The vector $\boldsymbol{x}_1$ has an arbitrary distribution.
3) For $m = 2, \cdots, M$, the real and imaginary parts of the components of $\boldsymbol{x}_m$ are independent zero mean unit variance Gaussian random variables.

Note that there is no loss of generality in assuming that $\boldsymbol{x}_1$ is the channel with arbitrary distribution because the ordering of the channels does not affect the GC estimate.

Using a Gram–Schmidt procedure, the vectors $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M$ can be expressed in terms of orthogonal vectors $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M$ as follows. For $2 \le m \le M$, let

$$\boldsymbol{y}_1 \doteq \boldsymbol{x}_1 \tag{1}$$

$$\boldsymbol{y}_m \doteq \boldsymbol{x}_m - P(\boldsymbol{x}_m | \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{m-1}) \tag{2}$$

where $P(\boldsymbol{x}_m | \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{m-1})$ denotes the orthogonal projection of $\boldsymbol{x}_m$ into the $(m-1)$-dimensional subspace of $\mathbb{C}^{\mathrm{M}}$ spanned by $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{m-1}$; i.e.

$$P(\boldsymbol{x}_m | \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{m-1}) \doteq \sum_{j=1}^{m-1} \alpha_{m,j} \boldsymbol{y}_j$$

with

$$\alpha_{m,j} = \begin{cases} \dfrac{\langle \boldsymbol{x}_m, \boldsymbol{y}_j \rangle}{\langle \boldsymbol{y}_j, \boldsymbol{y}_j \rangle} & \text{if } \|\boldsymbol{y}_j\| \ne 0 \\ 0 & \text{otherwise.} \end{cases}$$

Recursive use of (1) and (2) constructs $\boldsymbol{y}_m$ as a linear combination of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m$, and hence, it can be expressed as

$$\boldsymbol{y}_m = \sum_{k=1}^{m} \beta_{m,k} \cdot \boldsymbol{x}_k$$

where $\beta_{m,k}$ are complex scalars. As shown in [7], the Gram matrix determinant satisfies

$$g(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k + \lambda \cdot \boldsymbol{x}_j, \cdots, \boldsymbol{x}_M) = g(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M)$$

for any complex number λ and $j \ne k$, and

$$g(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M) = g(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M).$$

Hence, the GC estimate can be expressed as

$$\hat{\gamma}^2_{M,N}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M) = 1 - \frac{g(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M)}{\|\boldsymbol{x}_1\|^2 \cdots \|\boldsymbol{x}_M\|^2} \hat{=} 1 - z. \tag{3}$$

However, $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M$ are orthogonal by design. Thus, $g(\boldsymbol{y}_1, \cdots, \boldsymbol{y}_M)$ is diagonal, and $z = z_2 \cdot z_3 \cdots z_M$ with

$$z_m \hat{=} \frac{\|\boldsymbol{y}_m\|^2}{\|\boldsymbol{x}_m\|^2}$$

for $2 \le m \le M$. The Pythagorean theorem implies

$$\|\boldsymbol{y}_m\|^2 = \|\boldsymbol{x}_m\|^2 - \sum_{j=1}^{m-1} \frac{|\langle \boldsymbol{x}_m, \boldsymbol{y}_j \rangle|^2}{\|\boldsymbol{y}_j\|^2} \tag{4}$$

and it follows that

$$z_m = 1 - \sum_{j=1}^{m-1} \frac{|\langle \boldsymbol{x}_m, \boldsymbol{y}_j \rangle|^2}{\|\boldsymbol{x}_m\|^2 \cdot \|\boldsymbol{y}_j\|^2} \tag{5}$$

Notice, however, that

$$\frac{|\langle \boldsymbol{x}_m, \boldsymbol{y}_j \rangle|^2}{\|\boldsymbol{x}_m\|^2 \cdot \|\boldsymbol{y}_j\|^2} \hat{=} \hat{\gamma}^2_{2,N}(\boldsymbol{x}_m, \boldsymbol{y}_j) \tag{6}$$

is the MSC estimate of $\boldsymbol{x}_m$ and $\boldsymbol{y}_j$. As shown in [4] and [5], if one of the two channels contains only zero mean Gaussian noise and the two channels are statistically independent of each other, then the probability distribution of the MSC estimate is unaffected by the distribution of the other channel. However, this is implicit in the hypotheses: In the formulation of each z_m, the $\boldsymbol{y}_j$ are linearly dependent on $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_j$, and the maximum value of j is $m-1$. Therefore, $\boldsymbol{x}_m$ and $\boldsymbol{y}_j$ are independent, and the GC estimate is unaffected by the distribution of one channel.

Finally, note that the expression

$$\hat{\gamma}^2_{M,N}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_M) = 1 - \prod_{m=2}^{M} \left[1 - \sum_{j=1}^{m-1} \hat{\gamma}^2_{2,N} \left(\boldsymbol{x}_m, \sum_{k=1}^{j} \beta_{j,k} \cdot \boldsymbol{x}_k \right) \right] \tag{7}$$

gives the GC estimate explicitly in terms of MSC estimates.

By using a result obtained in [5], it is possible to weaken the H_0 hypotheses for $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_M$. For invariance of the GC estimate to the distribution of one channel, it is sufficient if $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_M$ are statistically independent, and for $m = 2, \cdots, M$, the distributions of $\boldsymbol{x}_m$ are spherically symmetric.

III. Multiple-Channel Matched Filtering

One important application of the invariance result derived above is that it allows previously calculated detection thresholds for GC-based detectors (e.g., from [3]) to be used in situations where one channel is known not to contain only white Gaussian noise. In passive detection, this might occur if an interesting signal is observed in a spectrogram from one sensor. A GC-based detector can be used to correlate this "reference" channel with data from other sensors to seek a multisensor detection that will reveal information about the location of the signal source. Known detection thresholds will remain valid as long as the H_0 assumptions of this correspondence hold.

In active detection scenarios, an exact replica of the transmitted signal is typically available for use in detecting echo returns. Using such a replica as one of the channels in a GC detector yields a "multiple-channel matched filter" detector for which the correspondence between detection thresholds and false alarm probabilities is known under the H_0 assumptions of this correspondence. By determining sets of time lags at which the GC estimate formed from the sensor data and the transmitted signal replica exceeds detection threshold, one obtains a time difference of arrival (TDOA) map from which information about the location of objects can be deduced.

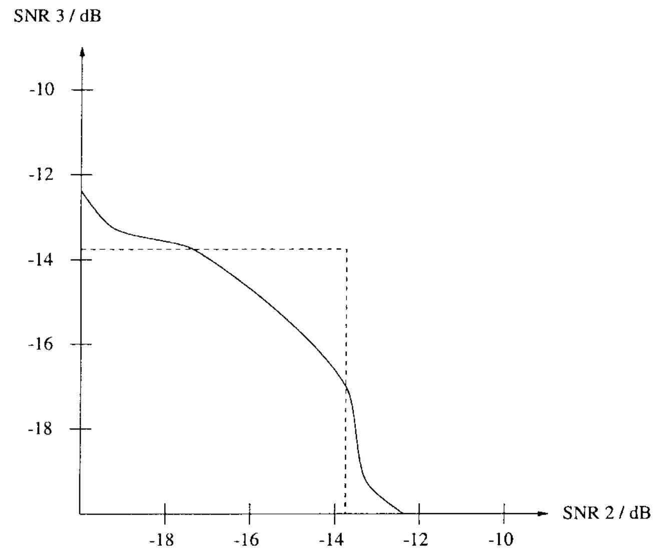

Fig. 1. Matched filter performance for a sinusoidal signal in white noise with $N = 256, P_f = 0.001$, and $P_d = 0.9$ (from simulations). The solid curve is the locus of SNR's on channels 2 and 3 necessary to achieve a detection with a three-channel GC detector having infinite SNR on channel 1. The dashed lines are the locus of SNR's on one channel to achieve a detection with a MSC detector having infinite SNR on channel 1.

Fig. 1 shows results obtained in a simulation of a three-channel matched filter scenario. The transmitted signal replica was represented by a complex white Gaussian vector $\boldsymbol{x}_1$ of length $N = 256$. The sensor data vectors $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ were obtained by adding independent white Gaussian noise vectors to $\boldsymbol{x}_1$, i.e., $\boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{n}_2$ and $\boldsymbol{x}_3 = \boldsymbol{x}_1 + \boldsymbol{n}_3$. During the simulation, the variances of the noise vectors $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ were adjusted to provide various signal-to-noise ratios (SNR's). Detection thresholds corresponding to a false alarm probability $P_f = 10^{-3}$ were chosen throughout the experiment.

A MSC detector was applied to the transmitted signal replica $\boldsymbol{x}_1$ and the data $\boldsymbol{x}_2$ from the first sensor. The SNR in $\boldsymbol{x}_2$ necessary to achieve a detection probability $P_d = 0.9$ was measured to be approximately -13.5 dB. This appears as a dashed vertical line in Fig. 1. The SNR in $\boldsymbol{x}_3$ necessary to achieve $P_d = 0.9$ was also measured and found to be approximately -13.5 dB. This appears as a dashed horizontal line in Fig. 1. Finally, a similar procedure was used to determine the locus of pairs of SNR's in $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ needed to achieve $P_d = 0.9$ with a three-channel GC detector. This locus is the solid curve in Fig. 1.

In the region of Fig. 1 to the left of the vertical dashed line, the SNR in $\boldsymbol{x}_2$ is too low for detection by an MSC detector using $\boldsymbol{x}_2$ and the transmitted signal replica $\boldsymbol{x}_1$. In the region below the horizontal dashed line, the SNR in $\boldsymbol{x}_3$ is too low for detection by an MSC detector using $\boldsymbol{x}_3$ and $\boldsymbol{x}_1$. It is interesting to note that some combinations of SNR's on the two sensor channels can be detected by the three-channel GC detector (i.e., are above and right of the solid locus), even though they are undetectable by either MSC detector.

To compute the GC estimate, it is necessary to calculate the inner products and norms of all channels. Since the MSC estimate is defined as the normalized inner product of two channels, the MSC estimates are automatically available if the GC estimate is computed. In some situations, it may be preferable to base detection of the GC estimate together with several MSC estimates. A more detailed description of such a strategy is given in [6].

IV. Discussion and Conclusions

The GC estimate has been shown to be invariant with respect to the statistical behavior of $\boldsymbol{x}_1$, provided that $\boldsymbol{x}_2, \cdots, \boldsymbol{x}_M$ have stationary Gaussian distributions and are statistically independent of $\boldsymbol{x}_1$. This invariance extends the utility of the GC estimate from passive to active detection scenarios. An example simulating a three-channel matched filter scenario was described. This example demonstrated that a GC-based multiple-channel matched filter can, at least in some cases, provide better detection performance than is obtained using multiple individual two-channel MSC-based detectors.

Other applications, in the area of cyclostationary signal detection as suggested by [8], for example, appear promising but need further investigation.

References

[1] G. C. Carter, C. Knapp, and A. H. Nuttall, "Estimation of the magnitude-squared coherence function via overlapped fast Fourier transform processing," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 331–344, Aug. 1973.

[2] H. Gish and D. Cochran, "Generalized coherence," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Apr. 1987, vol. 5, pp. 2745–2748.

[3] D. Cochran, H. Gish, and D. Sinno, "A geometric approach to multiple-channel signal detection," *IEEE Trans. Signal Processing*, vol. 43, pp. 2049–2057, Sept. 1995.

[4] A. H. Nuttall, "Invariance of distribution of coherence estimate to second-channel statistics," *IEEE Trans. Acoust, Speech, Signal Processing*, vol. ASSP-29, pp. 120–122, Feb. 1981.

[5] H. Gish and D. Cochran, "Invariance of the magnitude-squared coherence estimate with respect to second-channel statistics," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1774–1776, Dec. 1987.

[6] D. Sinno and D. Cochran, "Invariance of the generalized coherence estimate with respect to reference channel statistics," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Mar. 1992, vol. 2, pp. 505–508.

[7] P. J. Davis, *Interpolation and Approximation*. New York: Dover, 1975.

[8] S. Enserink and D. Cochran, "On detection of cyclostationary signals," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, May 1995, vol. 3, pp. 2004–2007.

On the Equivalence of the Operator and Kernel Methods for Joint Distributions of Arbitrary Variables

Akbar M. Sayeed

***Abstract*—Generalizing the concept of time-frequency representations, Cohen has recently proposed a method, based on operator correspondence rules, for generating joint distributions of arbitrary variables. As an alternative to considering all such rules, which is a practical impossibility in general, Cohen has proposed the kernel method in which different distributions are generated from a fixed rule via an arbitrary kernel. In this correspondence, we derive a simple but rather stringent necessary condition, on the underlying operators, for the kernel method (with the kernel functionally independent of the variables) to generate *all* bilinear distributions. Of the specific pairs of variables that have been studied, essentially only time and frequency satisfy the condition; in particular, the important variables of time and scale do not. The results warrant further study for a systematic characterization of bilinear distributions in Cohen's method.**

I. Introduction

Time-frequency representations (TFR's), such as the Wigner distribution and the short-time Fourier transform, represent signal characteristics jointly in terms of time and frequency and are powerful tools for nonstationary signal analysis and processing [1]. However, due to their inherent structure, TFR's can accurately represent only a limited class of nonstationary signal characteristics. In an effort to expand the applicability of joint signal representations to a broader class of signals, substantial amount of research has been directed to the study of joint distributions of variables other than time and frequency [2]–[7]. Spurred by the interest in the wavelet transform [8], joint time-scale representations constituted the first such generalizations [2], [3] and have received considerable attention.

In view of this recent trend, general theories for joint distributions of arbitrary variables have been proposed by many authors [1], [5], [9]–[11]. The first such generalization was proposed by Scully and Cohen [12] and developed by Cohen [1], [5] in direct extension of his original method for generating joint TFR's [13]. Baraniuk proposed a general approach based on group theoretic arguments [9] that was shown by Sayeed and Jones [14], [15] to be equivalent to Scully and Cohen's method. Other covariance-based generalizations have also been proposed [10], [11], [16], which complement Cohen's distributional method by characterizing joint representations in terms of covariance properties. However, Cohen's method seems to be the most general approach to date since no joint group structure is imposed on the variables as is done in [10], [11], and [16].

Fundamental to Cohen's method is the idea of associating variables with Hermitian (self-adjoint) operators [1]. For given variables, the entire class of joint distributions is generated by the infinitely many (in general) operator correspondence rules for an exponential function of the variables (the characteristic function *operator method* [1]). As an alternative to considering all possible correspondence rules, which is a practical impossibility in general, Cohen has proposed the *kernel method* in which a fixed operator correspondence is used and different

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The author was with the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801 USA. He is now with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77005 USA.

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