A Sampling Approach to Bandlimiting Instantaneously Companded Signals

Jennifer Q. Trelewicz *
Ross Martin
Douglas Cochran
Telecommunications Research Center
Arizona State University
Tempe, AZ 85287-7206 USA

Abstract

A well-known theorem of Beurling demonstrates that distinct bandlimited signals remain distinct after instantaneous companding and ideal lowpass filtering. Motivated by this result, Landau developed an iterative method for recovering a bandlimited signal after companding and lowpass filtering. This paper describes an alternative to lowpass filtering for restoring the original bandwidth to a companded bandlimited signal prior to transmission on a bandlimited channel. This approach allows the original signal to be recovered without iteration, but it also introduces some synchronization issues. These issues are examined and the design of companding functions that endow the transmitted with desired characteristics is considered.

1. Introduction

In [3], Landau studies the problem of recovering a bandlimited signal after it has been instantaneously companded by composition with a monotone function and subsequently passed through an ideal lowpass filter to restore its original bandwidth. An elegant proof that Landau attributes to A. Beurling shows that such recovery is possible, at least in principle, by demonstrating that distinct bandlimited signals remain distinct after the companding and lowpass operations are undertaken. The remainder of [3] is concerned with developing an algorithm to accomplish the recovery, which turns out to be tractable using a non-trivial iterative method. The paper takes the perspective that the lowpass operation is imparted by the channel itself rather than by the transmitter prior to sending the signal. Consequently, it does not address the issue that a companded signal which is intentionally passed through a lowpass filter matching the channel bandwidth prior to transmission may be altered by the bandlimiting operation in such a way that the goals of the compander design are no longer achieved.

This paper is concerned with the utility of instantaneous companding for imparting desired characteristics to a signal that is to be transmitted through a bandlimited channel. It begins by introducing the following alternative approach to lowpass filtering for re-imposing the original bandwidth to a bandlimited signal after instantaneous companding which leads to a non-iterative reconstruction formula.

Put

$$S(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Let $f : \mathbf{R} \to \mathbf{C}$ be bandlimited to the band $[-\Omega, \Omega]$ in the sense that its Fourier transform $\hat{f}$ satisfies $\hat{f}(\omega) = 0$ for all $|\omega| > \Omega$. The Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem [4] implies

$$f(t) = \sum_{k \in \mathbf{Z}} f(kT) S(\Omega(t - kT)) \tag{1}$$

where $T = \pi/\Omega$. Knowledge of the samples $\{f(kT) : k \in \mathbf{Z}\}$ is thus equivalent to knowledge of f.

If f is "instantaneously companded" by composition with a monotone increasing function $\gamma : \mathbf{R} \to \mathbf{R}$, the resulting signal $\gamma \circ f$ is generally not bandlimited. But, with $\varphi : \mathbf{R} \to \mathbf{C}$ defined by

$$\varphi(t) = \sum_{k \in \mathbf{Z}} [\gamma \circ f](kT) S(\Omega(t - kT)) \tag{2}$$

φ is bandlimited to $[-\Omega, \Omega]$ and uniquely determines both f and $\gamma \circ f$:

$$f(t) = \sum_{k \in \mathbf{Z}} f(kT) S(\Omega(t - kT)) \tag{3}$$

*Please address correspondence regarding this paper to J. Q. Trelewicz at the address given or jentre@asu.edu.

$$= \sum_{k\in\mathbf{Z}} \gamma^{-1}(\gamma(f(kT)))S(\Omega(t-kT))$$
$$= \sum_{k\in\mathbf{Z}} \gamma^{-1}(\varphi(kT))S(\Omega(t-kT)) \quad (4)$$

Because the original signal f is directly recoverable from φ, φ can serve as a "representative" for f in transmission over a bandlimited channel. Heuristically, the mapping taking $\gamma \circ f$ to φ differs from the ideal low-pass operation considered by Landau in that the out-of-band components of $\gamma \circ f$ are allowed to alias into the band rather than being discarded. It is clear from (4) that this approach has the advantage that an iterative method is not required to recover f from φ.

Examination of the reconstruction formula (4) reveals that the simplicity of recovering the original signal f at the receiver compared to Landau's approach is achieved at the cost of introducing a potentially delicate synchronization issue. Specifically, the receiver must sample φ to reconstruct f and the sampling instants kT used by the receiver in the reconstruction formula (4) must exactly match those used by the transmitter in the synthesis formula (2) to obtain an accurate reconstruction of f.

However, the receiver may employ a second-order feedback synchronization loop technique to attempt accurate reconstruction in the presence of this sensitivity. The ℓ^2 norm of the samples may be used as the loop error criterion. Such a synchronization loop technique, along with classes of γ and f for which the technique may be applied, are proposed and evaluated using computer simulations. Additionally, the effect of channel noise on the reconstruction is simulated and evaluated.

2. Characteristics of φ

Because φ is the signal actually transmitted, the goal of compander design is to endow φ rather than the companded signal $\gamma \circ f$ itself with desired amplitude characteristics.

As a first step toward controlling the amplitude characteristics of φ, it is important that φ have finite energy when f has finite energy and that φ be bounded when the samples $[\gamma \circ f](kT)$ are bounded.

Consider first whether φ has finite energy. It is easy to construct companders γ for which $f \in \mathbf{L}^2$ does not imply that $\varphi \in \mathbf{L}^2$. It is necessary to restrict the choice of γ to eliminate these cases. The following theorem accomplishes this goal:

Theorem: Let f, γ, and φ be as defined above. Then $f \in \mathbf{L}^2$ implies $\varphi \in \mathbf{L}^2$ if there is a constant $c > 0$ such that $|\gamma(x)| \leq c|x|$ for all $x \in \mathbf{R}$.

Proof: $f(t) \in \mathbf{L}^2$ implies $f(kT) \in \ell^2$ by energy conservation of sampling.

It follows that $\gamma(f(kT)) \in \ell^2$ by the following argument:

$$\textstyle\sum_k |\gamma(f(kT))|^2 \leq \sum_k |c|f(kT)||^2 = c\,\|f(kT)\|^2 < \infty$$

Hence $\gamma(f(kT)) \in \ell^2$, which implies that $\varphi(t) \in \mathbf{L}^2$ by energy conservation during sampling. □

Next, consider whether φ is bounded for bounded samples $\gamma(f(kT))$. This is important, since choice of γ directly manipulates the amplitude characteristic of the samples $\gamma(f(kT))$, but control of the amplitude characteristic of φ is what is actually desired. Thus, some link between the two amplitude characteristics must be established for this technique to make theoretical sense. The following example shows that a bound on the samples cannot be turned into a worst-case bound on φ without some additional knowledge:

Example: Consider a sequence $a_k \in \ell^\infty$ whose values are defined by

$$a_k = \begin{cases} -1, & S\left(\frac{\pi}{T}\left(\frac{T}{2} - kT\right)\right) < 0 \\ 1, & \text{else} \end{cases}$$

Define φ by

$$\varphi(t) = \sum_{k\in\mathbf{Z}} a_k S\left(\frac{\pi}{T}(t-kT)\right)$$

Then

$$\varphi\left(\frac{T}{2}\right) = \sum_{k\in\mathbf{Z}} a_k S\left(\frac{\pi}{T}\left(\frac{T}{2} - kT\right)\right)$$
$$= \sum_{k\in\mathbf{Z}} \left|S\left(\frac{\pi}{T}\left(\frac{T}{2} - kT\right)\right)\right| = \sum_{k\in\mathbf{Z}} \left|\frac{1}{\frac{\pi}{2} - k\pi}\right|$$

This sum, comparable to $\sum_{k=0}^{\infty} \frac{1}{k}$, clearly diverges. □

The example shows that a bounded sequence of samples can reconstruct to a function that is of arbitrarily large amplitude.

At this time, the authors are working on further results in this area. In particular, formulation of the problem in terms of Nemytskij operators ([1],[2]) allows the use of theorems establishing conditions under which such operators on $\mathbf{L}^p$ have range contained in $\mathbf{L}^q$.

3. Two Sample Companders

Consider $f \in \mathbf{L}^1 \cap \mathbf{L}^2$. (Simulations were performed with a normally-distributed random signal f taking values in $[-2, 2]$.) This section discusses two companders used in simulations that follow.

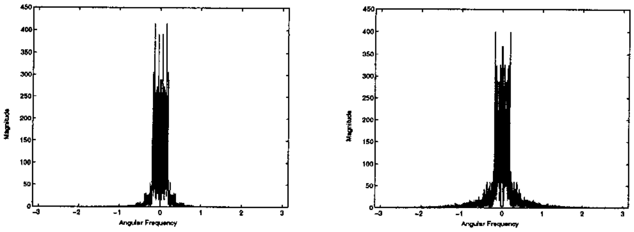

Figure 1. Magnitude Spectra of $\gamma_\mu(f)$ (left) and $\gamma_s(f)$ (right)

Define γ_μ, the μ-law compander with parameters μ, μ' (in the common version, $\mu = 255$ and $\mu' = \mu/\|f\|_\infty$):

$$\gamma_\mu(f) = \frac{\ln(1+\mu'|f|)}{\ln(1+\mu)} \operatorname{sgn} f$$

Put $A = \frac{1}{\ln(1+\mu)}$. Note that γ_μ is smooth and invertible in f. The left side of figure 1 shows the magnitude spectrum of $\gamma_\mu(f)$. The magnitude spectrum matches that of f only in the width of the main lobe: f lacks the sidebands seen on $\gamma_\mu(f)$.

It is also interesting to consider a compander that, although not necessarily practical, produces results seemingly less attractive than those of the μ-law compander. Define the following:

$$\gamma_s(f) = \begin{cases} f^2 \operatorname{sgn}(f), & |f| \leq 1 \\ |f|^{1/4} \operatorname{sgn}(f), & |f| \geq 1 \end{cases}$$

Note that

- $\gamma_s(f)$ is continuous and invertible, but not smooth.
- Although the bandlimitedness of $\gamma_s(f)$ is suggested by 1, Paley-Wiener considerations imply $\gamma_s(f)$ may not be bandlimited for some f.

(It is not as easy to conclude under what conditions $\varphi_s \in \mathbf{L}^2$.) The right side of figure 1 shows the magnitude spectrum of $\gamma_s(f)$. The magnitude spectrum matches that of f only in the width of the main lobe: f lacks the sidebands seen on $\gamma_s(f)$. Note that the band of $\gamma_s(f)$ extends to the observation band.

The results on Nemytskij operators can be used to say more about φ_μ and φ_s; however, such results are not necessary to illuminate the simulations below and thus will be reported as a part of continuing study of the method.

4. Reconstruction

As mentioned in section 1, perfect reconstruction of f through the reconstruction formula (4) requires perfect synchronization of the receiver with the transmitter.

This section proposes and evaluates some methods of employing a second-order feedback synchronization loop technique to attempt accurate reconstruction of the information signal.

4.1. Characteristics of the Reconstructed Signal

Suppose that the receiver samples with period T' and phase error ϵ. Call the reconstructed signal f'. Note that

$$\begin{aligned} \varphi(kT'+\epsilon) &= \sum_{m\in\mathbf{Z}} \gamma(f(mT))S(\Omega(kT'+\epsilon-mT)) \\ f'(t) &= \sum_{k\in\mathbf{Z}} \gamma^{-1}(\varphi(kT'+\epsilon))S(\Omega(t-kT')) \end{aligned}$$

Therefore, if ϵ and T' are small, $\varphi(kT'+\epsilon)$ is "close to" $\varphi(kT)$. If γ^{-1} is continuous, $f'(t)$ will then be "close to" $f(t)$. Therefore, it is desired that γ (and thus γ^{-1}) be continuous. (Both companders considered in section 3 above are indeed continuous.)

4.2. Reconstruction by Synchronization Signal

One method that may be used for recovering the signal involves the use of a predetermined synchronization ("sync") signal. In a multiple-channel system such as a satellite, where one channel may be reserved for a sync signal, synchronization can be performed continuously for the recovery of sampling phase.

Alternatively, the sync signal may be included in the data channel as a periodic burst ("frame sync"), followed by companded and aliased data. This method is useful in those systems where the receiver loop does not drift appreciably over one "frame" time (i.e., in the time between training signals).

The sync signal may be designed with the desired channel characteristics, thus allowing transmission without companding. Because the literature contains much research on the subject of "optimal" sync signals, and because the accuracy of phase reconstruction is dependent on the sync signal, such a system was not simulated here.

4.3. Reconstruction by Carrier Recovery

Another method for signal recovery involves modulating f on a low-frequency carrier during companding.

(Note that $f(t)\cos(\omega_c t+\theta) \in \mathbf{L}^2$.) The bandwidth of φ is then modified to take this carrier into account.

Choose $\omega_c \geq 2\Omega$. Put $T_c = \frac{\pi}{\omega_c+\Omega}$. Then

$$\varphi(t) = \sum_{k\in\mathbf{Z}} \gamma(f(kT_c)\cos(\omega_c kT_c+\theta))S(\Omega(t-kT_c))$$

The frequency spectrum of the reconstructed signal f' is examined for the characteristic of f: a bandpass signal from $[\omega_c-\Omega, \omega_c+\Omega]$ (positive frequency). Additionally, the $\mathbf{L}^2$ norm of the out-of-band (OOB) signal can be used as an error signal for the recovery loop: by the nature of (4), f' will be bandlimited to $[-\omega_c-\Omega, \omega_c+\Omega]$, but f has no energy in the band $[-\omega_c+\Omega, \omega_c-\Omega]$.

Suppose that the signal is sampled at the rate T_c', with phase error ϵ. Assume that the post processor uses samples of the signal for reconstruction. Consider the OOB energy for use as a loop error signal, Assume that the receiver analyzes a block of 2^N samples at the $\frac{\pi}{\omega_c+\Omega}$ rate, and that the FFT is used to find the signal spectrum (giving 2^N spectral samples). Then samples may be indexed by $k \in \{0,...,2^N-1\}$ to get frequencies $\pm 2\Omega$.

Suppose that $\omega_c = 2\Omega$. Then γ must be chosen so that

$$\lim_{T_c'\to T_c, \epsilon\to 0} \sum_{k=-2^{N-1}}^{2^{N-1}-1} |\widehat{f'}(k)|^2 = 0$$

Additionally, because of frequency error in the receiver, the integration may actually be performed over $[-\omega_c'+\Omega', \omega_c'-\Omega']$, where $\Omega' = \frac{\pi}{T_c'}$ and $\omega_c' \neq \omega_c$. The compander γ should also be chosen so that the error signal is robust to perturbations in the limits of integration.

One disadvantage of this approach is the higher sampling rate required for φ because of the carrier. If the method is implemented in a DSP, higher power dissipation will be realized in the system as a result. Additionally, modulation followed by the companding and aliasing operation may cause φ to have (positive and negative) bandwidth $2(\omega_c+\Omega)$. Since bandwidth is a valuable commodity on many communications channels, the resulting bandwidth expansion can make this method impractical for those channels. However, although the communications rate per unit bandwidth is adversely affected, the absolute communications rate (e.g., bits per second) is maintained in the system.

Furthermore, since the error depends on the out-of-band energy, relatively high-amplitude broadband noise can have a negative impact on the convergence of the recovery loop by imposing a floor on the out-of-band energy when sampling rate and phase are locked with the transmitter. (However, simulations discussed below show that wideband noise may have a smaller impact than may be expected.)

The fraction of the energy that was in the band $[-\Omega, \Omega]$ by sampling phase error was simulated for φ_μ and for φ_s. The simulations showed that in the absence of channel noise, the error signal is monotonic increasing with distance from the zero-offset phase, modulo the sampling interval. Therefore, the OOB energy is a reasonable candidate for a loop error signal.

One method of characterizing the second-order loop used to acquire the data is simulation of the effect of phase and frequency errors on the multiplicative separation between the minimum two OOB energy values. If there is little separation between the values, then very high gain will be required to discern distinct values, possibly causing the loop to become unstable.

The simulated loop used a threshold (around 1% for high SNR) before the loop was considered to be in lock. If the loop used no threshold, but instead only searched for the lowest error value, the worst case phase error at the correct sampling frequency sometimes caused the loop to "lock" to the incorrect sampling frequency.

Figures 2 and 3 show the ratios of the minimum two energy values for a sweep of phase and frequency errors, respectively. The solid line corresponds to γ_μ, and the dashed line to γ_s. (Phase errors are permitted to vary over every possible discrete value. Frequency errors are permitted to vary between half and double the true sampling frequency, non-inclusive.) Where the ratio is off of the plot, the minimum error value was not always at the correct value. The values off of the plot are used to indicate that the loop would have been unable to converge in this situation.

It can be seen from the figures that the wider-band γ_s gives energy ratios that continue to increase with SNR, implying that a system using γ_s might be optimized for faster loop convergence than one using γ_μ. Additionally, the γ_s system failed at a lower SNR than did the γ_μ system.

4.4. Reconstruction by Oversampling

Some of the problems inherent to the carrier recovery method can be addressed through the following method. Suppose that the receiver samples at the interval T/M, where $M > 1$. Therefore, a perfectly reconstructed signal will be bandlimited to $\pm\Omega$. Furthermore, if reconstruction phase is incorrect, the reconstructed signal will be bandlimited to $\pm 2\Omega$. Hence, the ℓ^2 norm of the energy in the $[\Omega, 2\Omega]$ band may be used as a loop error signal as in the carrier recovery algorithm.

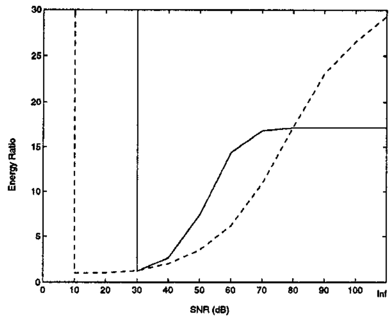

Figure 2. OOB Energy Ratio by SNR for Phase Error Only

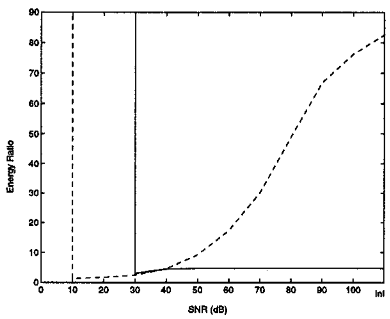

Figure 3. OOB Energy Ratio by SNR for Frequency Error

It should be noted that the oversampling algorithm is essentially equivalent to the carrier recovery algorithm. Therefore, no additional simulation results are shown here.

The oversampling algorithm does not require the doubling of the bandwidth, as is required by the carrier recovery algorithm. However, the oversampling algorithm has the same possible DSP power dissipation problems encountered in the carrier recovery method. Furthermore, it is possible that the loop error function may have more than one global minimum, in terms of T', depending on the sampling rate used. If the loop converges to the wrong minimum, the energy outside the $\pm\Omega$ band will be zero, but the error becomes independent of the sampling phase. For some classes of companders, this shortcoming can be addressed by choosing $\Omega > \Omega_f$, where f is bandlimited to Ω_f and the loop is not permitted to sample slower than $\frac{\pi}{\Omega_f}$. (More details of this problem will be discussed at a later time.)

5. Conclusions

In simulation, the sampling approach to bandlimiting an instantaneously companded signal was found to allow reconstruction of the original signal. Some of the difficulties introduced by the phase-lock requirements of the receiver were discussed, and a possible solution was simulated. Results of the phase-lock simulation were encouraging for medium-to-high SNR, when the OOB energy was considered.

Further work in this direction will develop the theoretical bounds, so that a class of candidate companders can be defined. Early simulation results suggest that wider-bandwidth companders might allow for better receiver performance and lower recovered signal error rates. The research can then be extended to find companders γ from which specific amplitude distributions in φ may be constructed.

References

[1] J. Appell and P. P. Zabrejko. *Nonlinear Superposition Operators.* Cambridge University Press, New York, 1990.

[2] V. Khatskevich and D. Shoiykhet. *Differentiable Operators and Nonlinear Equations.* Birkhäuser Verlag, Boston, 1994.

[3] H. J. Landau. On the recovery of a band-limited signal after instantaneous companding and subsequent band limiting. *Bell System Technical Journal,* XXXIX(2):351–363, March 1960.

[4] R. J. Marks. *Introduction to Shannon Sampling and Interpolation Theory.* Springer-Verlag, New York, 1991.

[5] W. Rudin. *Real and Complex Analysis.* McGraw-Hill, New York, 1987.

[6] B. Sklar. *Digital Communications: Fundamentals and Applications.* P T R Prentice-Hall, Englewood Cliffs, NJ, 1988.