Characterization of Non-Uniformly Spaced Discrete-Time Signals from Their Fourier Phase

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Abstract

Much work has been published on characterization and reconstruction of signals from their Fourier phase and from nonuniformly spaced sample values. In certain applications, the only available measurements of a signal are nonuniformly spaced samples whose Fourier magnitude information is unavailable or known to be corrupted. This paper gives an alternate proof to Shitz's and Zeevi's theorem on unique characterization of nonuniformly spaced discrete-time signals from Fourier phase. They used Logan's theorem on unique determination of a real, band-pass signal from its zero-crossings. This paper employs the more direct approach of representing signals as Weierstrass canonical products. This representation reveals the underlying structure of the signals and illuminates the effects of the hypotheses on them.

1. Introduction

Two problems of longstanding interest in signal processing are reconstruction of continuous-time signals from samples and reconstruction of continuous-time, finite-energy signals from their Fourier phase. Both of these fall into the category of "ill-posed inverse problems" since continuous-time signals are not (in general) uniquely characterized by either sample values or partial Fourier transform information. Regularization conditions (e.g., bandlimitedness, minimum phase) can be imposed to reduce the space of signals under consideration to the point that the mapping from the reduced space to the collection of possible sample sequences or phase functions is injective. This paper extends results of M. Hayes [1] on unique characterization of real-valued, discrete-time signals from their Fourier phase to the setting in which such sequences arise from the nonuniform sampling of continuous-time signals. Although this problem was solved by Shitz and Zeevi [2] using Logan's theorem [3] on unique determination of a real, bandpass signal from its zero-crossings, the results presented here were derived independently and the approach is more direct. In our approach, signals are represented as Weierstrass canonical products. This representation reveals the underlying structure of the signals and more readily exhibits the effect of each hypothesis on them. Hadamard's theorem relating the order and genus of an entire function is used to justify this representation. The main result gives explicit conditions under which discrete-time signals are characterized up to a positive factor by their Fourier phase. If the conditions are met, the sampling sequence is (in principle, at least) recoverable from its phase. In this case, recoverability of the original continuous-time signal from the sampling sequence implies its recoverability from the phase.

Section 2 presents preliminary information while the main result is presented in section 3. Section 4 draws the analogy between Hayes' result and those derived in section 3. It also provides a partial time-domain description of signals that satisfy the hypotheses of theorem 3.1. Theorem 3.1 is modified in section 5 to include complex discrete-time signals.

2. Preliminaries

Let f(z) be an entire function with zeros $\{u_n\}_{n=0}^{\infty}$ (including multiple zeros). Then the order of $f(z)$ is a number λ such that

$$\lambda = \limsup_{r\to\infty} \frac{\ln(\ln(M(r)))}{\ln(r)}$$

where

$$M(r) = \max\{|f(z)| : |z| = r \geq 0\}$$

The genus of f(z) is the smallest nonnegative integer, h, such that

$$\sum_{n=0}^{\infty} 1/|u_n|^{h+1} < \infty$$

The following factorization theorem for entire functions is due to Weierstrass.

Theorem 2.1 (Weierstrass) *Every entire function, $f(z)$, that is not identically zero can be represented in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) e^{p_n(z)}$$

where the product is taken over all the zeros of $f(z)$ other than $z = 0$, m is a nonnegative integer, $g(z)$ is an entire function, and the polynomials $p_n(z)$ are given by

$$p_n(z) = \sum_{k=0}^{\infty} \frac{1}{k}\left(\frac{z}{z_n}\right)^k \quad (n = 1, 2, \ldots) \quad \blacksquare$$

One version Hadamard's theorem is as follows:

Theorem 2.2 (Hadamard) *If $f(z)$ is a nonzero, entire function of finite order, then $g(z)$ in the Weierstrass factorization of $f(z)$ above is a polynomial of degree no larger than the order of $f(z)$, and the degree of each $p_n(z)$ can be set equal to the genus of $f(z)$.* $\blacksquare$

If $f(z)$ has finite order, then the Weierstrass factorization resulting from applying Hadamard's theorem to $f(z)$ is called *canonical.*

3. The Theorem

Theorem 3.1 *Let $x(t) = \sum_{n=0}^{N} x_n \delta(t - t_n)$ and $y(t) = \sum_{m=0}^{M} y_m \delta(t - \tilde{t}_m)$. Define*

$$X(z) = \sum_{n=0}^{N} x_n e^{-it_n z} \quad \text{and} \quad Y(z) = \sum_{m=0}^{M} y_m e^{-i\tilde{t}_m z}$$

and suppose the following hypotheses:

1. *$arg(X(z)) = arg(Y(z)) \quad \forall z \in \mathbf{R}$.*
2. *$\overline{z}_N \neq 0$ is not a zero of $X(z)$ whenever z_N is a zero of $X(z)$ and similarly $\overline{w}_N \neq 0$ is not a zero of $Y(z)$ whenever w_N is a zero of $Y(z)$.*
3. *$\tilde{t}_m, t_n, y_m, x_n \in \mathbf{R}$, $m = 0, \ldots, M$ and $n = 0, \ldots, N$.*

   It then follows that $y(t) = Ax(t)$ where $A \geq 0$.

Proof:

The theorem is proven by showing that the entire functions $X(z)$ and $Y(z)$ are of order less than or equal to one. Hadamard's theorem implies that $X(z)$ and $Y(z)$ have Weierstrass canonical factorizations. Finally, the Weierstrass canonical factorizations of $X(z)$ and $Y(z)$ are shown to differ by a real, positive constant. Note that $X(z)$ and $Y(z)$ are the Fourier transforms of the signals $x(t)$ and $y(t)$ respectively.

The order of $X(z)$ is bounded as follows:

$$|X(z)| \leq \sum_{n=0}^{N} |x_n| e^{|t_n||z|} \leq \sum_{n=0}^{N} |x_n| e^{T|z|}$$

where $T = \max\{|t_n| : 0 \leq n \leq N\}$, hence

$$\ln[M(r)] \leq \ln\left[\sum_{n=0}^{N} |x_n| e^{Tr}\right] \leq 2Tr$$

for large r. Therefore

$$\limsup_{r\to\infty} \frac{\ln(\ln(M(r)))}{\ln(r)} \leq \lim_{r\to\infty} \frac{\ln(2T) + \ln(r)}{\ln(r)} = 1.$$

Replacing $X(z)$ with $Y(z)$ in the calculation above shows that the order of $Y(z)$ is also less than or equal to one.

Hadamard's theorem now applies yielding the following Weierstrass canonical factorizations of $X(z)$ and $Y(z)$:

$$X(z) = z^m e^{a+bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) e^{\alpha z/z_n}$$

and

$$Y(z) = z^l e^{c+dz} \prod_{n=1}^{\infty} (1 - \frac{z}{w_n}) e^{\beta z/w_n}$$

where

$$a, b, c, d \in \mathbf{C};\ m, l \in \{0, 1, 2, \ldots\};$$
$$\alpha, \beta \in \{0, 1\};\ \text{and } z_n, w_n \neq 0\ \forall n.$$

The values of α and β depend on the genus of $X(z)$ and $Y(z)$ respectively. Hence, if $\sum_{n=0}^{\infty} 1/|z_n| < \infty$ then

$\alpha = 0$, otherwise $\alpha = 1$. Similarly, if $\sum_{n=0}^{\infty} 1/|w_n| < \infty$, then $\beta = 0$, otherwise $\beta = 1$.

Hypothesis 1 implies that $arg(X(z)/z^m) = arg(Y(z)/z^m)$ and $arg(X(z)/z^l) = arg(Y(z)/z^l)\,\forall z \in \mathbf{R}$, but $arg\{X(z)/z^m\}$ and $arg\{Y(z)/z^l\}$ are defined for $z = 0$; hence, $l = m$.

Hypothesis 1 also implies that

$$\frac{X(z)}{Y(z)} = \overline{\left(\frac{X(z)}{Y(z)}\right)} = \frac{\overline{X(\overline{z})}}{\overline{Y(\overline{z})}}\ \forall z \in \mathbf{R}$$

where an overbar indicates complex conjugation. Since $\overline{X(\overline{z})}$ and $\overline{Y(\overline{z})}$ are entire functions, $\overline{Y(\overline{z})}\,X(z)$ and $\overline{X(\overline{z})}\,Y(z)$ are both entire and equal $\forall z \in \mathbf{R}$ so they are equal $\forall z \in \mathbf{C}$. Therefore, the equation

$$X(z)\overline{Y(\overline{z})} = \overline{X(\overline{z})}Y(z)$$

reduces to

$$e^{a+bz+\overline{c}+\overline{d}z} \prod_{n=1}^{\infty}\left((1-\frac{z}{z_n})(1-\frac{z}{\overline{w}_n})e^{\alpha z/z_n+\beta z/\overline{w}_n}\right) =$$
$$e^{\overline{a}+\overline{b}z+c+dz} \prod_{n=1}^{\infty}\left((1-\frac{z}{\overline{z}_n})(1-\frac{z}{w_n})e^{\alpha z/\overline{z}_n+\beta z/w_n}\right) \quad (1)$$

There are several observations to make regarding this equation.

1. If z_N is a zero of the left-hand side (LHS) with a multiplicity of $r > 0$, then z_N ($= w_M$ for some M) with multiplicity s or $\overline{z}_N$ with multiplicity t are zeros of the right-hand side (RHS). Hypothesis 2 implies that $t = 0$. Equality is maintained if both sides of equation 1 are divided by $(z - z_N)^r$ or by $(z - z_N)^s$, hence $r = s$. A similar argument show that if w_M is a zero of the RHS with a multiplicity of h, then w_M ($= z_N$ for some N) is zero of the LHS with a multiplicity of h.

2. Recall that the values of α and β depend on the convergence properties of $\sum_{n=0}^{\infty} 1/|z_n|$ and $\sum_{n=0}^{\infty} 1/|w_n|$ respectively. Hypothesis 2 and observation 1 applied to equation 1 imply that $\{z_n\}_{n=1}^{\infty} = \{w_n\}_{n=1}^{\infty}$, hence $\alpha = \beta$.

3. Observations 1 and 2 show that the products in equation 1 are equal. Cancelling the products from both sides of equation 1 yields

$$e^{a+bz+\overline{c}+\overline{d}z} = e^{\overline{a}+\overline{b}z+c+dz} \quad (2)$$

which is valid for every z that is not a zero of equation 1, hence equation 2 is valid for all z. Equation 2 implies that

$$a + \overline{c} = \overline{a} + c + 2\pi ik \ \ ,k \in \mathbf{Z}$$

and

$$b + \overline{d} = \overline{b} + d$$

hence,

$$Im(a) - \pi k = Im(c) \quad and \quad Im(b) = Im(d)$$

4. Hypothesis 3 requires $x(t)$ and $y(t)$ to be real-valued functions. Using the definition of $X(z)$ given in the statement of the theorem and the real-valued nature of $x(t)$ and $y(t)$, one can show that $\overline{X(-\overline{z})} = X(z)$. Equating the canonical factorizations of $\overline{X(-\overline{z})}$ and $= X(z)$ yields

$$(-1)^m\, z^m e^{\overline{a}-\overline{b}z} \prod_{n=1}^{\infty}(1+\frac{z}{\overline{z}_n})e^{-\alpha z/\overline{z}_n} =$$
$$z^m e^{a+bz} \prod_{n=1}^{\infty}(1-\frac{z}{z_n})e^{\alpha z/z_n} \quad (3)$$

Note that if z_N is a zero of $X(z)$, then so is $-\overline{z}_N$. Cancelling common terms in equation 3 yields

$$(-1)^m\, e^{\overline{a}-\overline{b}z} = e^{a+bz} \implies$$
$$(-1)^m = e^{a-\overline{a}}e^{(b+\overline{b})z} \quad (4)$$

Setting $z = 0$ in equation 4 implies that

$$(-1)^m = (e^{i\pi})^m = e^{a-\overline{a}} = e^{2iIm(a)} \implies$$
$$Im(a) = \frac{\pi m}{2} + \pi j\,, \quad j \in \mathbf{Z}$$

Equation 4 also implies that $e^{(b+\overline{b})z}$ is a constant which gives $b + \overline{b} = 2Re(b) = 0$. Similarly, hypothesis 3 implies that $\overline{Y(-\overline{z})} = Y(z)$ which leads to $d + \overline{d} = Re(d) = 0$.

All of the essentials for proving the theorem are now in place. Observations 1 through 4 imply that $X(z)$ and $Y(z)$ can be written as follows:

$$X(z) = z^m e^{Re(a)+i(\frac{\pi}{2}m+\pi j)} e^{Im(b)z} \prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1+\frac{z}{\overline{z}_n})e^{\alpha z/z_n\ -\ \alpha z/\overline{z}_n}$$

and

$$Y(z) = z^m e^{Re(c)+i(\frac{\pi}{2}m+\pi j-\pi k)} e^{Im(b)z} \prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1+\frac{z}{\overline{z}_n})e^{\alpha z/z_n\ -\ \alpha z/\overline{z}_n}$$

Finally, $arg(X(z)) = arg(Y(z))\,\forall z \in \mathbf{R}$ implies that k is an even integer, hence $Y(z) = AX(z)$ where $A = e^{Re(c)-Re(a)} \geq 0$ and the theorem is proven. ∎

4. Discussion of the Results

A result analogous to theorem 3.1 was proven by Hayes [1] under the assumption of uniform spacing between consecutive sample times. In his development, Hayes used both the z-transform and a hypothesis excluding signals which possessed a nonzero complex number, w_0, and the reciprocal of its complex conjugate, $1/\overline{w}_0$, as zeros of the signal's z-transform. Theorem 3.1 employs both the Fourier transform and hypothesis 2 which excludes signals that possess a nonzero complex number, z_0, and its conjugate, $\overline{z}_0$, as zeros of the signals Fourier transform. Now w_0 and $1/\overline{w}_0$ are symmetric points with respect to the unit circle, and evaluation of the z-transform on the unit circle yields the discrete-time Fourier transform of the signal. Similarly, z_0 and $\overline{z}_0$ are symmetric points with respect to the real axis and evaluation of the Fourier transform on the real axis yields the discrete-time Fourier transform of a signal. Hence, Hayes' result and theorem 3.1 are analogous in that signals possessing zeros that are symmetric about the curve generating the discrete-time Fourier transform are not recoverable (up to a multiplicative factor) solely from their Fourier phase.

Clearly the class of functions satisfying theorem 3.1 is not empty. Direct calculations show that hypotheses 2 and 3 of theorem 3.1 are satisfied by any signal of the form $x(t) = x_0 + 2x_0\delta(t-t_1)$ where $x_0, t_1 \in \mathbf{R}$ and $x_0, t_0 \neq 0$. Note that the zeros of $X(z)$ are $z = (-i\ln(2) + \pi + 2\pi k)/t_1$ where $k \in \mathbf{Z}$.

It is interesting to note that functions which are symmetric about some time t_s do not meet hypothesis 2. This is shown as follows:

Let $x(t) = \sum_{n=0}^{N} x_n\delta(t_n)$ be a symmetric, real-valued signal with $t_m < t_n$ for $m < n$. Assume that $X(z_0) = 0$, then

$$\begin{aligned} 0 &= X(z_0) = \sum_{n=0}^{N} x_n e^{-it_n z_0} \\ &= e^{-it_N z_0}\sum_{n=0}^{N} x_n e^{i(t_N - t_n)z_0} = \overline{X(z_0)} \\ &= e^{it_N \overline{z}_0}\sum_{n=0}^{N} x_n e^{-i(t_N-t_n)\overline{z}_0} \\ &= e^{i(t_N+t_0)\overline{z}_0} X(\overline{z}_0) \end{aligned}$$

The last equality follows from the symmetry of $x(t)$ which implies that $t_N - t_n = t_{N-n} - t_0$ and $x_n = x_{N-n}$; hence, $\bar{z}_0$ is also a zero of $X(z)$. The class of signals that do not meet hypothesis 2 contains more than the symmetric, real-valued signals since any non symmetric, real-valued signal containing a real zero also violates hypothesis 2. An example is $x(t) = x_0 - x_0\delta(t-t_1)$ where $x_0, t_1 \in \mathbf{R}$ and $x_0, t_1 \neq 0$. This signal is not symmetric and possesses a zero at $2\pi/t_1$. Further research is needed in order to render complete time-domain descriptions of signals which meet and violate the hypotheses of theorem 3.1.

5. Extending To Complex Signals

Modifying hypothesis 3 of Theorem 3.1 yields the following theorem applicable to complex-valued discrete-time signals:

Theorem 5.1 *Let $f(t) = \sum_{n=0}^{N} f_n\delta(t-t_n)$ and $g(t) = \sum_{n=0}^{N} g_n\delta(t-t_n)$. Define*

$$F(z) = \sum_{n=0}^{N} f_n e^{-it_n z} \quad and \quad G(z) = \sum_{n=0}^{N} g_n e^{-it_n z}$$

and suppose the following hypotheses:

1. *$arg(F(z)) = arg(G(z)) \quad \forall z \in \mathbf{R}$*
2. *$\overline{z}_N \neq 0$ is not a zero of $F(z)$ whenever z_N is a zero of $F(z)$ and similarly $\overline{w}_N \neq 0$ is not a zero of $G(z)$ whenever w_N is a zero of $G(z)$*
3. *$t_n \in \mathbf{R}; x_n, y_n \in \mathbf{C}$ and $n = 0, \ldots, N$*

   It then follows that $g(t) = Ae^{(Re(d)-Re(b))z}f(t)$ where $A \geq 0$.

Proof:

Since the first two hypotheses of theorems 3.1 and 5.1 are the same, the analysis of theorem 3.1 up through observation 3 apply yielding

$$F(z) = z^m e^{Re(a)+iIm(a)+(Re(b)+iIm(b))z} \prod_{n=1}^{\infty}(1-\frac{z}{z_n})e^{\alpha z/z_n}$$

and

$$G(z) = z^m e^{Re(c)+i(Im(a)-\pi k)+(Re(d)+iIm(b))z} \prod_{n=1}^{\infty}(1-\frac{z}{z_n})e^{\alpha z/z_n}$$

where $k \in \mathbf{Z}$. When $x(t)$ and $y(t)$ are complex-valued functions no further analysis is possible except to note that $arg(F(z)) = arg(G(z))\ \forall z \in \mathbf{R}$ implies that k is an even integer. Thus, $g(t) = Ae^{(Re(d)-Re(b))z}f(t)$ where $A = e^{Re(c)-Re(a)} \geq 0$ and the theorem is proven. ∎

6. Summary

In this paper, we developed conditions under which nonuniformly spaced real-valued discrete-time signals are uniquely characterized up to a positive factor. We also developed conditions under which nonuniformly spaced complex-value discrete-time signals are uniquely characterized up to an exponential function. Our approach utilized the Weierstrass canonical factorization of the signals. This representation revealed the underlying structure of the signals and illuminated the effects of the hypotheses on them.

References

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