GENERALIZED WAVELET TRANSFORMS AND THE CORTEX TRANSFORM

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ABSTRACT

This paper presents a generalization of the continuous wavelet transform to higher dimensions. This generalization includes previous higher dimensional generalizations as special cases, without the loss of the elegance of the one-dimensional equations. One noteworthy special case is a transform that is closely related to the cortex transform developed by Watson. This special case firmly relates the cortex transform to wavelet theory. This paper emphasizes the utility of the theory by using it to derive an implementation of the cortex transform.

1. Introduction

Currently, there are several known multidimensional generalizations of the one-dimensional wavelet transform. In the most widely known generalization, one-dimensional transforms are applied separately in each orthogonal direction. This is referred to herein as the *rectilinear* wavelet transform, and it is the method used by most wavelet-based image compression techniques.

Another wavelet generalization is the two dimensional transform by Murenzi[1], literature on which can be more easily found in the paper by Antoine *et al.* [2]. This is referred to herein as the *circular* wavelet transform, since it generalizes the dilations of the wavelet transform to the set of dilations and rotations on $\mathbb{R}^2$.

Neither of these generalizations encompasses the other. Thus, a question of interest is whether there is a stronger generalization that encompasses both. This paper answers this question by presenting a generalized wavelet transform that includes as special cases the one-dimensional, rectilinear, and circular wavelet transforms.

Particular attention is given to the circular wavelet transform, which is shown to be a result of the relationship between $\mathbb{R}^2$ and $\mathbb{C}$ and is further shown to be strongly related to the cortex transform of Watson [3]. An implementation of the cortex transform from the circular wavelet transform is developed as an example of this relationship.

2. One-Dimensional Wavelet Transform

This section presents the essentials of one-dimensional wavelet analysis for comparison with the generalizations in the following sections. Details and proofs can be found in [4] or [5].

Let $a, b, x \in \mathbb{R}$, $h \in L^2(\mathbb{R})$, and let $\hat{h}$ be the Fourier transform of h. Define

$$\left[\Delta^{(a,b)}h\right](x) = \frac{1}{\sqrt{|a|}}\, h\left(\frac{x-b}{a}\right)$$

and

$$C_h = \int_{\mathbb{R}} \frac{|\hat{h}(\omega)|^2}{|\omega|} d\omega.$$

If C_h is finite, h is *admissible* as a mother wavelet. In this case, the reconstruction theorem states that any $f \in L^2(\mathbb{R})$ can be reconstructed from its wavelet coefficients $\langle f, \Delta^{(a,b)}h\rangle$ by the formula

$$f = \frac{1}{C_h}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{a^2}\left\langle f, \Delta^{(a,b)}h\right\rangle \Delta^{(a,b)}h \; db\, da.$$

The reconstruction theorem is one of the most essential features of wavelet analysis.

3. Rectilinear Wavelet Transform

The rectilinear wavelet transform generalizes the one-dimensional transform to a separable multidimensional transform by letting $a, b, x, \omega \in \mathbb{R}^n$, *i.e.*

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix},$$

with the following alterations in the definitions and theorem:

$$\left[\Delta^{(a,b)}h\right](x) = \frac{1}{\sqrt{|a_1 a_2 \cdots a_n|}}\, h\left(\begin{bmatrix} \frac{x_1-b_1}{a_1} \\ \frac{x_2-b_2}{a_2} \\ \vdots \\ \frac{x_n-b_n}{a_n} \end{bmatrix}\right)$$

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$$C_h = \int_{\mathbb{R}^n} \frac{|\hat{h}(\omega)|^2}{|\omega_1\omega_2\cdots\omega_n|} d\omega$$

$$f = \frac{1}{C_h}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{a_1^2 a_2^2 \cdots a_n^2} \left\langle f, \Delta^{(a,b)}h \right\rangle \Delta^{(a,b)}h \, db\, da.$$

This generalization is equivalent to taking the one-dimensional wavelet transform successively in each orthogonal direction of the multidimensional space. For $n = 1$, the rectilinear wavelet transform reduces to the one-dimensional wavelet transform.

4. Circular Wavelet Transform

An alternative generalization of the one-dimensional wavelet transform to $\mathbb{R}^2$ is the circular wavelet transform [1, 2], for which $a \in \mathbb{R}$ is replaced with the pair $(a, \theta) \in (0, \infty) \times [0, 2\pi)$ and $b, x, \omega \in \mathbb{R}$ are replaced by $b = [b_1, b_2]^T \in \mathbb{R}^2$, $x = [x_1, x_2]^T \in \mathbb{R}^2$, and $\omega = [\omega_1, \omega_2]^T \in \mathbb{R}^2$.

Let r_θ denote the matrix which performs rotation in $\mathbb{R}^2$ by θ degrees. Then the definitions and theorem for circular wavelet analysis are

$$\left[\Delta^{(a,\theta,b)}h\right](x) = \frac{1}{a}\, h\left(a^{-1} r_{-\theta}(x - b)\right)$$

$$C_h = \int_{\mathbb{R}^2} \frac{|\hat{h}(\omega)|^2}{||\omega||^2} d\omega$$

$$f = \frac{1}{C_h}\int_{\mathbb{R}}\int_0^{2\pi}\int_{\mathbb{R}^2} \frac{1}{a^4} \left\langle f, \Delta^{(a,\theta,b)}h \right\rangle \Delta^{(a,\theta,b)}h \; a\, db\, d\theta\, da.$$

Clearly, the circular wavelet transform has a different character than the rectilinear wavelet transform on $\mathbb{R}^2$. In particular, it allows non-separable wavelets in two dimensions and replaces independent dilation in each dimension by rotation and dilation simultaneously in both dimensions.

5. Generalized Wavelet Transform

Despite their distinct characters, this section shows that both the rectilinear and circular wavelet transforms arise as special cases of a more general multidimensional wavelet transform, the formulation of which is based on *ring multiplications* on $\mathbb{R}^2$. The algebraic terminology used in this section is defined in [6].

Consider $\mathbb{R}^n$ as a vector space over the field $\mathbb{R}$. If a ring multiplication "$\circ$" is introduced to $\mathbb{R}^n$ so that $\mathbb{R}^n$ forms an *algebra* under the multiplication, then ring multiplication by a constant $a \in \mathbb{R}^n$ is, by the definition of an algebra, a linear operation on $\mathbb{R}^n$. Thus, for each $a \in \mathbb{R}^n$ there is a linear operator $L_a : \mathbb{R}^n \rightarrow \mathbb{R}^n$ called the *left regular representation* of a and defined by $L_a x = a \circ x$. There is an associated linear operator $S_x : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $S_\omega a = L_a^T \omega$. Suppose the ring multiplication has the property that the set of elements with ring inverses are dense in $\mathbb{R}^n$. Then, with $h \in L^2(\mathbb{R}^n)$ and $a, b, x, \omega \in \mathbb{R}^n$, the following definitions and theorem form the mathematical foundation of a generalized multidimensional wavelet analysis:

$$\left[\Delta^{(a,b)}h\right](x) = \frac{1}{\sqrt{|\det L_a|}} h(L_a^{-1}(x - b)),$$

$$C_h = \int_{\mathbb{R}^n} \frac{|\hat{h}(\omega)|^2}{|\det S_\omega|} d\omega.$$

$$f = \frac{1}{C_h}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{|\det L_a|^2} \left\langle f, \Delta^{(a,b)}h \right\rangle \Delta^{(a,b)}h \; db\, da.$$

As examples of this, suppose the ring multiplication on $\mathbb{R}^n$ is defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \circ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_n b_n \end{bmatrix}$$

then L_a and S_a are the diagonal matrices defined by $L_a = S_a = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. This makes the generalized definitions and theorem equivalent to the definitions and theorem of rectilinear wavelet analysis.

As pointed out above, rectilinear wavelet analysis with $n = 1$ is the same as one-dimensional wavelet analysis. Thus both one-dimensional wavelet analysis and rectilinear wavelet analysis arise as special cases of this formulation.

Similarly, circular wavelet analysis comes from the ring multiplication "$\circ$" on $\mathbb{R}^2$ defined by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \circ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 - a_2 b_2 \\ a_1 b_2 + a_2 b_1 \end{bmatrix}.$$

This ring multiplication is same as the multiplication of the complex numbers. In this case, equating a with the complex number $a_1 + a_2 i$ to take its absolute value and angle,

$$\begin{aligned} L_a &= \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} = |a| r_{\angle a} \\ S_x &= \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}, \end{aligned}$$

and the generalized definitions and reconstruction theorem become equivalent to the definitions and reconstruction theorem of circular wavelet analysis.

6. Cortex Transform

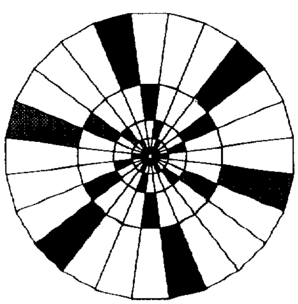
Circular wavelet analysis has fundamental variables of scale (which corresponds to logarithmic frequency), rotation, and two-dimensional positional shift. These

Figure 1: The Leena image and the result after it has been decomposed and reconstructed by the transform.

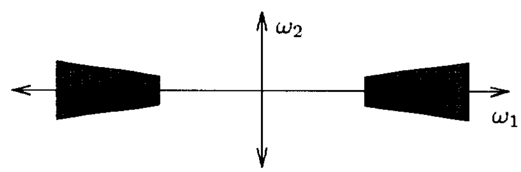
correspond to the basic organization of Watson's cortex transform[3], which was devised as a means to process images in a similar way to the processing in the human visual cortex. Hence there is a relationship between the circular wavelet transform, the cortex transform, and image processing in the human visual cortex. It is shown here that this relationship can be strengthened by actually deriving the cortex transform from the circular wavelet transform.

The cortex transform was implemented by Watson by partitioning the frequency domain into separate angularly oriented regions at logarithmically spaced frequency steps, bandlimiting the image to each region, and sampling the bandlimited pieces as efficiently as possible using available techniques from sampling theory.

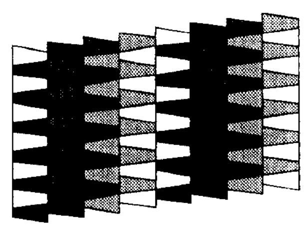
Following a similar procedure, the frequency domain can be divided according to the figure below, where zero frequency is in the center:



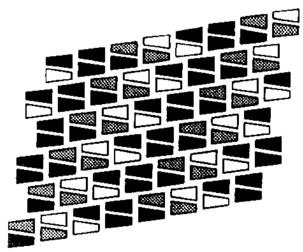
This division corresponds to a discrete sampling of the frequency and rotation variables of the circular wavelet transform. In order to bandlimit the image to a given region, the "mother wavelet" h is chosen so that its Fourier transform $\hat{h}$ is the characteristic function a *bin*, which consists of two opposing frequency regions as shown below:



Once the image is bandlimited to a bin, positional samples are used to represent the bandlimited function. These positional samples produce frequency replications. Aliasing is avoided by preventing these frequency replications from overlapping. An example of an optimal sampling for a bin is shown below.



This frequency replication pattern is according to theory the optimally efficient replication that avoids aliasing, but in practice aliasing is not avoided due to filter imperfections. Thus, it is often desirable to introduce a controlled amount of inefficiency in order to reduce aliasing. In this case, the frequency replication pattern looks more like this:



The efficiency of this sampling is controlled, so it is easy to make it more efficient than the fixed efficiency sampling proposed by Watson [3].

Preliminary results show that the transform adequately represents an image and reconstructs it from its representation. These results are presented in Figure 1. In order to obtain these results, special consideration must be paid to preserving frequencies near zero, since the transform has a singularity there. It is also crucial that the image be of relatively large size (at least 256×256 pixels), since the processing assumes an image is continuous, and the violation of this assumption by a coarsely sampled image introduces significant artifacts.

7. Discussion and Conclusions

The generalized wavelet analysis introduced in this paper encompasses three important special cases of wavelet analysis: one-dimensional wavelet analysis, rectilinear wavelet analysis, and circular wavelet analysis. Additional ring structures on $\mathbb{R}^n$ lead to other wavelet analyses, applications for which are as yet unexplored.

Preliminary results presented in the preceding section demonstrate that the wavelet-based cortex transform adequately represents an image and reconstructs it from its representation, provided some care is taken in the implementation. This transform implementation could be derived entirely from sampling theory, as Watson did in his original derivations. However, the close ties to wavelet theory add additional perspective to the problem, and in particular they show that reconstruction should be possible with different mother wavelets h. This fact is of particular interest for this problem, since the original idea was to process signals in ways analogous to the processing in the visual cortex. Choice of h to mimic neural receptive fields is one approach that might further this goal.

Further work is suggested in investigating special cases of the generalized wavelet transform. There are many special cases other than those presented here, some of which may have properties that would make them useful in applications.

Further work is also suggested in modifying the implementation of the circular wavelet transform to use different mother wavelets h that achieve closer ties with human visual physiology.

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9. References

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