A Cyclostationary Feature Detector

Scott Enserink

Telecommunications Research Center
Arizona State University
Tempe, AZ 85287-7206

Douglas Cochran

Department of Electrical Engineering
Arizona State University
Tempe, AZ 85287-7206

Abstract

Cyclostationary models for communications signals have been shown in recent years to offer many advantages over stationary models. Stationary models are adequate in many situations, but they cause important features of the signal to be overlooked. One such important feature is the correlation between spectral components that many signals exhibit. Cyclostationary models allow this spectral correlation to be exploited.

This paper presents a signal detector that exploits spectral correlation to determine the presence or absence of a cyclostationary signal in noise. The detector's probability of false alarm is analytically derived. Computer simulations verify that the analytical derivation is correct. The detector's receiver operating characteristic curves are determined from the simulation data and the analytical expression for the probability of false alarm.

1 Introduction

Within the past few years, numerous processing techniques that exploit cyclostationary features of signals have been introduced. Among the recent developments in this area is a simple *single-cycle* detector for cyclostationary signals proposed by Gardner [1]. This paper introduces a modified version of Gardner's single-cycle detector in which magnitude-squared coherence (MSC) estimation is used as a measure of the spectral correlation in a signal. When no spectral correlation (and hence no cyclostationary component) is present at a particular frequency and cyclic rate, the distribution function of the MSC estimate used in the detector can be determined analytically. Therefore, the behavior of this detector in a noise-only (H_0) environment can be determined and detection threshold values corresponding to desired false alarm probabilities can be computed. Computer simulations were performed to 1) verify the theoretical results and to 2) determine the receiver operating characteristics of the detector.

*This work was supported in part by Motorola Inc. under the University Partnerships in Research Program and in part by the U.S. Air Force under grant number F449620-93-1-0051. The support of an Armed Forces Communications and Electronics Association Post-Graduate fellowship is also gratefully acknowledged.

2 Mathematical Preliminaries

The purpose of this preliminary section is to introduce notation and terminology that will be used throughout the remaining sections of this article. The mathematical development for cyclostationary analysis set forth in this section is based on Gardner's work, further descriptions of which can be found in [2].

A stochastic process is said to be wide-sense cyclostationary if the following equations hold for the mean, m_X, and the autocorrelation function, R_X, of X

$$m_X(t+kT) = m_X(t) \tag{1}$$

$$R_X(t_1+kT, t_2+kT) = R_X(t_1, t_2) \tag{2}$$

Equation (2) can be re-written as

$$R_X(t+kT+\frac{\tau}{2}, t+kT-\frac{\tau}{2}) = R_X(t+\frac{\tau}{2}, t-\frac{\tau}{2}) \tag{3}$$

This equation shows that if X is cyclostationary, then, for a given value of τ, the autocorrelation function, R_X, is periodic with respect to t with period T. Thus, for a fixed τ, R_X can be expressed as a Fourier series

$$R_X(t+\frac{\tau}{2}, t-\frac{\tau}{2}) = \sum_{\alpha} R_X^{\alpha}(\tau)e^{i2\pi\alpha t} \tag{4}$$

1058-6393/95 $4.00 © 1995 IEEE

In this series the Fourier coefficients R_X^α are defined as

$$R_X^\alpha(\tau) \stackrel{\text{def}}{=} \lim_{Z\to\infty} \frac{1}{Z}\int_{-\frac{Z}{2}}^{\frac{Z}{2}} R_X(t+\frac{\tau}{2}, t-\frac{\tau}{2})\, e^{-i2\pi\alpha t}\, dt \quad (5)$$

By using the limit as Z goes to infinity, the expression allows for the presence of multiple periodicities in the signal that are not harmonically related. The sum in (4) is taken over all integer multiples of $1/T$. That is $\alpha = k/T$ for all $k \in Z$. The value $1/T$ is referred to as the fundamental frequency. A process X is said to *exhibit cyclostationarity* if there exists an α for which the Fourier coefficient defined by (5) is nonzero. The function R_X^α defined by (5) is called the *cyclic autocorrelation function*, and α is called the *cyclic frequency* parameter.

3 Spectral Correlation Density Function

One of the fundamental concepts of cyclostationary analysis is that certain spectral components of cyclostationary signals are correlated. This fact is the basis for the detector that is described later in this paper.

The spectral correlation present in a cyclostationary signal is measured by the spectral correlation density (SCD) function. The SCD of a process X is defined as the Fourier transform of the cyclic autocorrelation function,

$$S_X^\alpha(f) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} R_X^\alpha(\tau) e^{-i2\pi f\tau}\, d\tau \quad (6)$$

The spectral correlation density can be defined in an alternative but equivalent manner from the viewpoint of spectral correlation. Before the alternative definition is given some notation must be defined. The Fourier transform of the segment of X on the bounded time interval $[t - W/2, t + W/2]$ is defined as

$$\tilde{X}_W(t,v) \stackrel{\text{def}}{=} \int_{t-\frac{W}{2}}^{t+\frac{W}{2}} X(u) e^{-i2\pi uv}\, du$$

The time-averaged correlation of the two spectral components with frequencies $v = f + \alpha/2$ and $u = f - \alpha/2$, normalized by the length of the time interval of the finite segment, is

$$\lim_{Z\to\infty} \frac{1}{Z}\int_{-\frac{Z}{2}}^{\frac{Z}{2}} \frac{1}{W} E[\tilde{X}_W(t, f+\frac{\alpha}{2})\, \tilde{X}_W^*(t, f-\frac{\alpha}{2})]dt \quad (7)$$

With this notation, the cyclic spectral density can be defined as the limit of the spectral correlation in (7) as the spectral resolution $\Delta f = 1/W$ becomes infinitesimal:

$$S_X^\alpha(f) \stackrel{\text{def}}{=} \lim_{W,Z\to\infty} \frac{1}{Z}\int_{-\frac{Z}{2}}^{\frac{Z}{2}} \frac{1}{W} E[\tilde{X}_W(t, f+\frac{\alpha}{2})\tilde{X}_W^*(t, f-\frac{\alpha}{2})]dt \quad (8)$$

By use of the convolution theorem, this expression can be shown to be equivalent to (6).

4 Spectral Autocoherence Function

The spectral autocoherence function is a type of correlation coefficient that measures the degree of correlation between two spectral components of a random process. The spectral autocoherence function is a special case of the coherence function. Estimators of the magnitude-squared of the coherence function (MSC estimators) have been well studied. For two wide-sense stationary (WSS) process X and Y the coherence function γ is defined as the normalized cross spectral density,

$$\gamma_{XY}(f) = \frac{S_{XY}(f)}{[S_X(f)S_Y(f)]^{\frac{1}{2}}} \quad (9)$$

Consider U and V, two frequency-shifted versions of X defined by

$$U(t) \stackrel{\text{def}}{=} X(t)e^{-i\pi\alpha t} \quad (10)$$

$$V(t) \stackrel{\text{def}}{=} X(t)e^{+i\pi\alpha t} \quad (11)$$

The spectral autocoherence function ρ_X^α comes from applying the definition of the coherence function to the processes U and V. The term *auto* is used because U and V are spectrally shifted versions of the same process.

$$\begin{aligned} \rho_X^\alpha(f) &= \frac{< S_{UV} > (f)}{[< S_U > (f) < S_V > (f)]^{\frac{1}{2}}} \\ &= \frac{S_X^\alpha(f)}{[< S_X > (f+\frac{\alpha}{2}) < S_X > (f-\frac{\alpha}{2})]^{\frac{1}{2}}} \end{aligned} \quad (12)$$

A process X is said to show complete coherence at spectral frequency f and cyclic frequency α if $|\rho_X^\alpha(f)| = 1$. A process X is said to be completely incoherent at spectral frequency f and cyclic frequency α if $|\rho_X^\alpha(f)| = 0$.

The detector presented in the next section is based on the above concepts. Its detection statistic is an estimate of the magnitude squared of the spectral autocoherence function $|\rho_X^\alpha(f)|^2$. The cumulative distribution function of the detection statistic will be determined in section 6 from previous studies of the cumulative distribution function of MSC estimates.

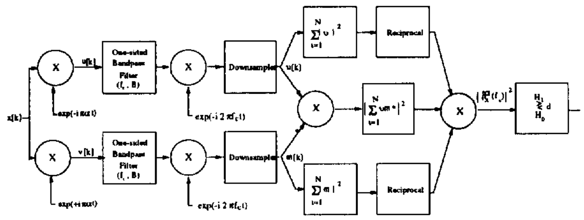

Figure 1: Discrete-time version of cyclostationary feature detector.

5 Description of the Detector

The cyclostationary feature detector described in this section determines the presence or absence of a signal having spectral coherence at a spectral frequency f and cyclic frequency α. For a given f, it detects spectral correlation at only one cyclic frequency α, thus it is also called a *single-cycle* detector.

The single-cycle detector is shown in figure 1. The spectral shifters and one-sided bandpass filters produce two spectral components of the process X. The first component is centered at frequency $f = f_c + \alpha/2$ and has bandwidth B, the bandwidth of the one-sided bandpass filter. The second spectral component is centered at frequency $f = f_c - \alpha/2$. These two signals are downsampled and shifted to baseband. The middle leg of the detector correlates them over a time interval T seconds (N independent samples) long to produce an estimate $|\hat{S}_X^\alpha(f_c)|^2$ of the magnitude squared of the spectral correlation density function at f_c and α. This section is identical to the detector originally introduced by Gardner. The upper and lower legs were added to Gardner's detector in order to obtain estimates of the time-averaged spectra of X at $f = f_c + \alpha/2$ and $f = f_c - \alpha/2$, which are denoted as $< \hat{S}_X > (f_c + \alpha/2)$ and $< \hat{S}_X > (f_c - \alpha/2)$. The addition of the upper and lower legs results in a detection statistic which is an estimate of the magnitude squared of the autocoherence function. Equation (12) shows this clearly,

$$|\hat{\rho}_X^\alpha(f)|^2 = \frac{|\hat{S}_X^\alpha(f)|^2}{< \hat{S}_X > (f+\frac{\alpha}{2}) < \hat{S}_X > (f-\frac{\alpha}{2})} \quad (13)$$

From the definition of the spectral correlation density function given in equation (8), it can be seen that the estimate $\hat{\rho}_X^\alpha$ of the autocoherence function approaches its true value as the integration time T goes to infinity and as the bandwidth B of the bandpass filters becomes infinitesimally small; i.e.,

$$\lim_{B\to 0}\lim_{T\to\infty} \hat{\rho}_X^\alpha(f) = \rho_X^\alpha(f) \quad (14)$$

This shows that the detection statistic is a function of the length of the correlation interval T and the bandwidth B of the bandpass filters, this relationship can be measured by the number of independent samples N used in the correlation sums. The relation of the detection statistic to N is the subject of the next section.

6 Probability of False Alarm

The detection statistic, $|\hat{\rho}_X^\alpha|^2$, of the single-cycle detector is an estimate of the magnitude-squared coherence of the two signal sequences (ν and ϖ in figure 1) that are the outputs of the downsamplers. The cumulative distribution of $|\hat{\rho}_X^\alpha|^2$ can thus be drawn from results previously obtained in the study of MSC estimators.

Magnitude-squared coherence estimators have been well studied in connection with sonar and radar applications. Nuttall documented the statistics of the output of a MSC estimator for the case were the true MSC, $|\gamma|^2$, is zero. He found that if one of the random processes has a white Gaussian distribution, then the cumulative distribution function of the MSC estimate, $\xi = |\hat{\gamma}|^2$, will be [3]

$$F_{\underline{\xi}}(\xi) = 1 - (1-\xi)^{N-1} \quad (15)$$

where N is the number of samples used in calculating ξ. Nuttall showed that this result holds regardless of the type of random process on the other channel. Gish and Cochran [4] generalized this result by showing that it still holds even if the restriction that one of the channels be white Gaussian is replaced with the restriction that one of the channels have a spherically symmetric distribution.

The detection statistic $|\hat{\rho}_X^\alpha(f)|^2$ used in the single-cycle detector is an estimate of the MSC of ν and ϖ, the signals at the outputs of the downsamplers. The mathematical calculation made by this MSC estimator from N complex samples $\{x_n + iy_n\}$ of ν and N complex samples $\{u_n + iv_n\}$ of ϖ is,

$$|\hat{\gamma}|^2 = \frac{|\sum_{n=1}^N (x_n+iy_n)(u_n-iv_n)|^2}{\sum_{n=1}^N |x_n+iy_n|^2 \sum_{n=1}^N |u_n+iv_n|^2}, \quad \mathrm{N} \geq 2 \quad (16)$$

If only WSS noise is present at the input to the detector then the true value of the autocoherence function will be zero. This corresponds to the situation for the MSC estimator where the true value of the MSC is $|\gamma|^2 = 0$. If the noise at the input is white Gaussian, then the noise after the bandpass filters will be

P_{FA}	N				
	8	16	32	64	128
10^{-1}	0.280	0.142	0.0716	0.0359	0.0180
10^{-2}	0.482	0.264	0.138	0.0705	0.0356
10^{-3}	0.627	0.369	0.200	0.104	0.0529
10^{-4}	0.732	0.459	0.257	0.136	0.0700
10^{-5}	0.807	0.536	0.310	0.167	0.0867
10^{-6}	0.861	0.602	0.360	0.197	0.103

Table 1: Detection threshold values for various values of N.

Gaussian as well. It should be noted here that the output of the bandpass filters will be approximately Gaussian for a broad class of stochastic input signals [5, 6]. Although the noise at the outputs of the bandpass filters will be correlated, if the downsampling is done correctly the noise will be whitened once again. As shown above, the remaining section of the single-cycle detector is exactly an MSC estimator. The input to the MSC section of the detector is white Gaussian, therefore the results in [3, 4] can be applied. Thus, for the case where only white Gaussian noise is present at the input of the detector, the detection statistic of the detector, $\psi = |\hat{\rho}_X^{\alpha}|^2$, will have the following cumulative distribution function,

$$F_{\underline{\psi}}(\psi | H_0) = 1 - (1 - \psi)^{N-1} \qquad (17)$$

The probability of false alarm, P_{FA}, can be determined easily from the cumulative distribution function of the detection statistic for the H_0 case. Let the threshold of the detector be d, then P_{FA} is given by,

$$\begin{aligned} P_{FA} &= Pr(\psi > d | H_0) \\ &= (1-d)^{N-1} \qquad (18) \end{aligned}$$

The threshold values that will give specific P_{FA}'s for various values of N are given in Table 1.

7 Simulations and Results

Computer simulations were performed to 1) verify the analytical expressions, given in equations (17) and (18), for the performance of the detector for the H_0 case, and 2) to determine the receiver operating characteristic (ROC) curves for the detector. For the H_0 case, white Gaussian noise was input to the detector. Curves were generated for various values of N, the number of independent samples used in the summations. A graph of the measured cumulative distributions is given in figure 2. The theoretical distributions

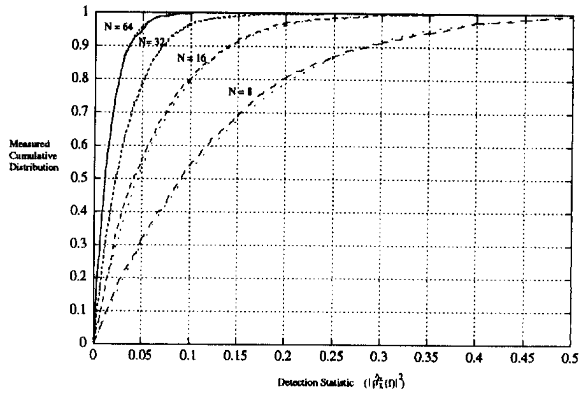

Figure 2: Measured and theoretical cumulative distributions of the detection statistic for various values of N for the H_0 case.

are also shown for the sake of comparison. It can be seen that the theoretical and empirical results match very closely. From these results it can be concluded that the theoretical analysis was accurate and that equation (17) provides a valid representations of the cumulative distribution of the detection statistic and that (18) provides a valid representation of the probability of false alarm for the single-cycle detector.

Following the verification of the theoretical analysis of the probability of false alarm, simulations were performed to measure the ROC curves for the single-cycle detector with a cyclostationary signal present. A binary-phase-shift-keyed (BPSK) signal was used as the cyclostationary signal. The spectral components centered about the carrier frequency, f_0, and separated by and amount $\alpha = R$, where R is the data rate, have complete correlation. The single-cycle detector was set to these values. At these values the autocoherence function for the BPSK signal has unity magnitude. For these simulations N was kept at 32 and the bandwidth of the bandpass filters was set to $0.075R$. The resulting ROC curves are shown Fig. 3. From these figure it can be seen that the detector, with the settings just described, performs well for SNR's in excess of 5dB, adequately for SNR's in the range of 0dB to 3dB, and poorly for SNR's below 0dB. The performance can be improved by decreasing the bandwidth of the detector's single-sided bandpass filters. It was shown that decreasing the bandwidth 15 times (from $0.075R$ to $0.005R$) led to an improvement in performance of approximately $11dB$, as was expected since $10\log(15) \approx 11dB$. Also, using larger values for N will improve the performance. The price paid for the improvement in performance in both of these cases is that a longer collection time is required.

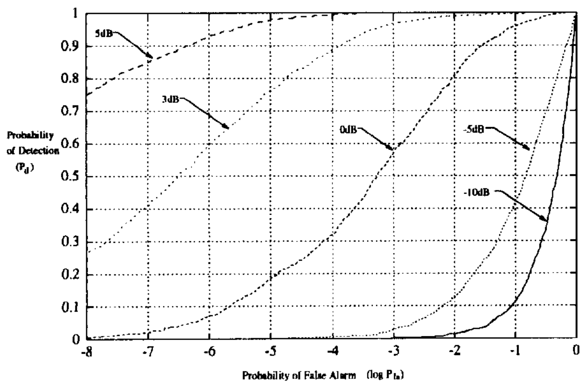

Figure 3: Measured receiver operating characteristics (-10dB to 5dB.)

In other simulations, a strong signal that had spectral coherence at a (f, α) pair other than at the one to which the detector was set was input to the detector. The measured cumulative distribution curves showed that the detector's performance for this situation was similar to its performance for the noise-only case. This indicates that the single-cycle detector could be used to determine the absence or presence of a certain cyclostationary signal in an environment that is filled with interfering signals, so long as none of the interferers was spectrally coherent at the same (f, α) pair as the signal of interest.

Additional simulations were run with a direct sequence spread spectrum (DS SS) signal and with a scale division multiple access (SDMA) signal. The results of these simulations showed the single-cycle detector is viable for detecting signals with either of these modulation types.

8 Conclusions

The detection statistic ($\psi = |\hat{\rho}_x^\alpha|^2$) of the proposed cyclostationary feature detector is an estimate of the magnitude squared of the spectral autocoherence function. Theoretical analysis has shown that the cumulative distribution of ψ is a function of the number of samples N used in calculating ψ and that for the H_0 case with Gaussian noise it will be

$$F_{\underline{\psi}}(\psi | H_0) = 1 - (1 - \psi)^{N-1}$$

Computer simulations of the detector's performance verified the validity of this equation. The probability of false alarm (P_{FA}) for a given threshold d was determined to be

$$P_{FA} = (1 - d)^{N-1}$$

The receiver operating characteristic curves were determined from the analytical expression for P_{FA} and the data from the computer simulations. The simulations showed that the detector is viable for detecting BPSK, DS SS, and SDMA signals and that its performance can be improved by increasing the collection time. The detector's simulated performance also showed that it is able to distinguish between cyclostationary signals that have different cyclostationary components (e.g., different cyclic frequencies α).

References

[1] W.A. Gardner, "Exploitation of spectral redundancy in cyclostationary signals," *IEEE Signal Processing Magazine*, vol. 8, no. 2, pp. 14-36, April 1991.

[2] W.A. Gardner, *Introduction to Random Processes with Application to Signals and Systems.* New York: McGraw-Hill, 1990.

[3] A.H. Nuttall, "Invariance of distribution of coherence estimate to second-channel statistics," *IEEE Transactions of Acoustics, Speech, and Signal Processing*, vol. ASSP-29, no. 1, pp. 120-122, February 1981.

[4] H. Gish and D. Cochran, "Invariance of magnitude-squared coherence estimate with respect to second-channel statistics," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-35, no. 12, pp. 1774-1776, December 1987.

[5] M. Rosenblatt, "Some comments on band-pass filters," *Quarterly of Applied Mathematics*, vol. 18, pp. 387-393, 1961.

[6] A. Papoulis, "Narrow-band systems and gaussianity," *IEEE Transactions on Information Theory*, vol. IT-18(1), January 1972.