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Phase and Magnitude in Normalized Images

Douglas Cochran

Abstract—**Distortion due to complete loss or perturbation of phase or magnitude information is considered for energy normalized images. With certain statistical assumptions on the image structure, phase information is shown to be more crucial.**

I. Introduction

The relative importance of Fourier phase and magnitude in various classes of signals has been a recurring topic in signal processing literature for more than a decade [2], [4]–[6]. This paper considers the case of energy-normalized monochrome images.

Normalization of processed images to a predetermined energy before comparison is a common practice because most important visual features in an image are unaffected by moderate changes in its overall brightness. Examples include the positions and orientations of edges, region boundaries, variations in texture, and the relative brightnesses of regions. Indeed, the human visual system adjusts the brightness of its inputs by means of pupil response and other nonlinear adaptation mechanisms [1], [3].

This paper examines distortion in energy normalized images due to perturbation or complete loss of phase or magnitude information. With certain statistical assumptions on the image structure, phase information is shown to be more crucial.

II. Image Representation

This paper will consider images that are Lebesgue measurable finite energy ($\boldsymbol{L}^2$) functions supported in a bounded region of the plane. For notational simplicity in Fourier series expansion, the region of support will be assumed to be contained in the square $\boldsymbol{I}^2 = \boldsymbol{I} \times \boldsymbol{I}$ where $\boldsymbol{I}$ denotes the interval $[0, 2\pi]$. Given an image $f \in \boldsymbol{L}^2(\boldsymbol{I}^2)$, there is a corresponding set of complex Fourier coefficients defined by

$$a_{m,n} \triangleq \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(x,y) e^{-i(mx+ny)} dx\, dy$$

for all $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. The image f is represented by its Fourier coefficients in that the 2–D Fourier series

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} a_{m,n} e^{i(mx+ny)}$$

converges to $f(x,y)$ almost everywhere in $\boldsymbol{I}^2$.

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Writing $a_{m,n} = c_{m,n} e^{i\theta_{m,n}}$ yields a representation

$$f(x,y) = \sum_m \sum_n c_{m,n} e^{i(mx+ny+\theta_{m,n})}$$

in which the magnitude coefficients $c_{m,n} \geq 0$ and phase coefficients $\theta_{m,n} \in \boldsymbol{I}$ appear explicitly. It is thus possible to regard an image as a function of its magnitude and phase coefficients as follows:

Definition: Let $\boldsymbol{c} \triangleq \{c_{m,n} | m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\}$ be a set of nonnegative magnitude coefficients with

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_{m,n}^2 < \infty$$

and let $\Theta \triangleq \{\theta_{m,n} | m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\}$ be a set of phase angles with each $\theta_{m,n} \in \boldsymbol{I}$. $F(\boldsymbol{c}, \Theta)$ is the image with values

$$F(\boldsymbol{c},\Theta)(x,y) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_{m,n} e^{i(mx+ny+\theta_{m,n})}$$

(i.e., the 2–D complex signal on $\boldsymbol{I}^2$ with magnitude coefficients $\boldsymbol{c}$ and phase coefficients Θ).

The norm of $f \in \boldsymbol{L}^2(\boldsymbol{I}^2)$ is defined by

$$||f|| \triangleq \left(\int_0^{2\pi} \int_0^{2\pi} |f(x,y)|^2 dx\, dy \right)^{1/2}$$

and the square of this quantity is the energy in f. Defining an energy normalization operator $\mathcal{N} : \boldsymbol{L}^2(\boldsymbol{I}^2) \to \boldsymbol{L}^2(\boldsymbol{I}^2)$ by

$$\mathcal{N} f(x,y) \triangleq \frac{f(x,y)}{||f||}$$

it is evident that $\mathcal{N} f$ has unit energy.

Parseval's relation implies that the norm of an image $f = F(\boldsymbol{c}, \Theta)$ can be expressed in terms of its magnitude coefficients $c_{m,n}$ by

$$||f|| = \left(\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_{m,n}^2 \right)^{1/2}.$$

This leads to the expression

$$\mathcal{N} f(x,y) \triangleq \mathcal{N} F(\boldsymbol{c},\Theta)(x,y) = \frac{\sum\sum c_{m,n} e^{i(mx+ny+\theta_{m,n})}}{\left(\sum\sum c_{m,n}^2\right)^{1/2}} \quad (1)$$

for the values of $\mathcal{N} f$ in terms of the magnitude and phase coefficients of f.

III. Distortion Due To Perturbations in Magnitude and Phase

One approach to the question of significance of magnitude and phase in normalized images is to consider the relative effects of small changes in each type of coefficient on the value of the signal. Some care is needed to make the notion of a "small change" in each kind of coefficient precise and meaningfully comparable.

Equation (1) gives the values of the energy normalized version $\mathcal{N} F(\boldsymbol{c}, \Theta)$ of an image $f = F(\boldsymbol{c}, \Theta)$ in terms of its magnitude coefficients $c_{m,n}$ and phase coefficients $\Theta_{m,n}$. It is clear from this expression that many different magnitude coefficient sequences $\boldsymbol{c}$ yield the same normalized signal. In particular, if $\lambda > 0$ then $\mathcal{N} F(\lambda \boldsymbol{c}, \Theta) \equiv \mathcal{N} F(\boldsymbol{c}, \Theta)$ so that there is a continuous family (indexed by $\mathbb{R}_+$) of magnitude coefficient sequences $\boldsymbol{c}$ that give rise to the same normalized image $\mathcal{N} F(\boldsymbol{c}, \Theta)$.

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It is this representational ambiguity for normalized images that complicates the definition of "small changes" in the magnitude coefficients. If $\mathbf{C} = \lambda \mathbf{c}$ then

$$\frac{\partial \mathcal{N}F}{\partial c_{m,n}} = \frac{\partial \mathcal{N}F}{\partial C_{m,n}} \cdot \frac{\partial C_{m,n}}{\partial c_{m,n}} = \lambda \frac{\partial \mathcal{N}F}{\partial C_{m,n}}.$$

In other words, by choice of representation, the derivative of $\mathcal{N}F$ with respect to a magnitude coefficient can be made arbitrary up to sign.

In order to make a meaningful comparison of the effects of perturbations to magnitude and phase coefficients, define a two-dimensional sequence $\boldsymbol{b} = \{b_{m,n} | m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\}$ by $b_{m,n} \triangleq \log c_{m,n}$. An energy normalized image may be regarded as a function of its *log-magnitude coefficients* $\boldsymbol{b}$ by

$$\mathcal{N}F(\boldsymbol{b}, \Theta)(x,y) = \frac{\sum\sum e^{b_{m,n} + i(mx+ny+\theta_{m,n})}}{\left(\sum e^{2b_{m,n}}\right)^{1/2}} = \mathcal{N}F(\boldsymbol{c}, \Theta)(x,y) \tag{2}$$

without the representational ambiguity discussed above. Moreover, the mapping $\mu(r, \theta) = e^{r+i\theta}$ from rectangular to log-polar coordinates has the property that

$$\left|\frac{\partial \mu}{\partial r}(r,\theta)\right| = e^r = \left|\frac{\partial \mu}{\partial \theta}(r,\theta)\right|$$

so that the partial derivatives of a normalized signal with respect to its log-magnitude and phase coefficients are directly comparable.

Consider the normalized 2-D signal $\mathcal{N}F(\boldsymbol{b}, \Theta)$ with values given by (2). Its derivative with respect to a single phase coefficient $\theta_{j,k}$ is

$$\frac{\partial \mathcal{N}F(\boldsymbol{b}, \Theta)}{\partial \theta_{j,k}}(x,y) = \frac{i e^{b_{j,k}} e^{i(jx+ky+\theta_{j,k})}}{\left(\sum\sum e^{2b_{m,n}}\right)^{1/2}} = \frac{i c_{j,k} e^{i(jx+ky+\theta_{j,k})}}{\left(\sum\sum c_{m,n}^2\right)^{1/2}}$$

so that

$$\left|\frac{\partial \mathcal{N}F(\boldsymbol{b}, \Theta)}{\partial \theta_{j,k}}\right|^2 = \frac{c_{j,k}^2}{\sum\sum c_{m,n}^2} = \frac{c_{j,k}^2}{||f||^2}$$

where $||f|| = \left(\sum_{m,n} c_{m,n}^2\right)^{1/2}$.

The derivative of $\mathcal{N}F(\boldsymbol{b}, \Theta)$ with respect to a log-magnitude coefficient will be found at the bottom of this page. Assuming the $\theta_{j,k}$ are independent and uniform on $[0, 2\pi]$ in this expression yields the expectation

$$E\left|\frac{\partial \mathcal{N}F(\boldsymbol{b}, \Theta)}{\partial b_{j,k}}\right|^2 = \frac{c_{j,k}^2 ||f||^2 - c_{j,k}^4}{||f||^4}$$

so that

$$\frac{E\left|\frac{\partial \mathcal{N}F(\boldsymbol{b},\Theta)}{\partial b_{j,k}}\right|^2}{\left|\frac{\partial \mathcal{N}F(\boldsymbol{b},\Theta)}{\partial \theta_{j,k}}\right|^2} = 1 - \frac{c_{j,k}^2}{\sum\sum c_{m,n}^2} \leq 1 \tag{3}$$

and hence

$$E\left|\frac{\partial \mathcal{N}F(\boldsymbol{b}, \Theta)}{\partial b_{j,k}}\right|^2 \leq \left|\frac{\partial \mathcal{N}F(\boldsymbol{b}, \Theta)}{\partial \theta_{j,k}}\right|^2.$$

In this sense, the expected distortion introduced by a small perturbation of a magnitude coefficient is less the distortion introduced by an equivalently small perturbation of a phase coefficient. Equation (3) shows that the most significant distortion results when the phase of a Fourier coefficient having large magnitude is perturbed.

Fig. 1. Original image $f(x, y)$.

IV. Distortion Due to Loss of Phase or Magnitude Information

In [5], Oppenheim and Lim observed that complete loss of magnitude information is less detrimental to images than complete loss of phase information. The authors do not say whether any normalization was applied to the images before display. An experiment similar to theirs in which energy normalization is used is as follows. The 2–D discrete Fourier transform (DFT) of an image having values $f(x, y)$ is computed yielding Fourier coefficients $\hat{f}(\omega_1, \omega_2)$. Two new frequency-domain signals $\hat{p}$ and $\hat{m}$ having values

$$\hat{p}(\omega_1, \omega_2) \triangleq \frac{\hat{f}(\omega_1, \omega_2)}{|\hat{f}(\omega_1, \omega_2)|}$$

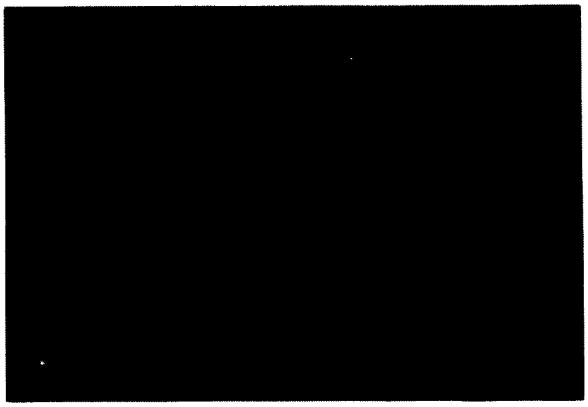
and

$$\hat{m}(\omega_1, \omega_2) \triangleq |\hat{f}(\omega_1, \omega_2)|$$

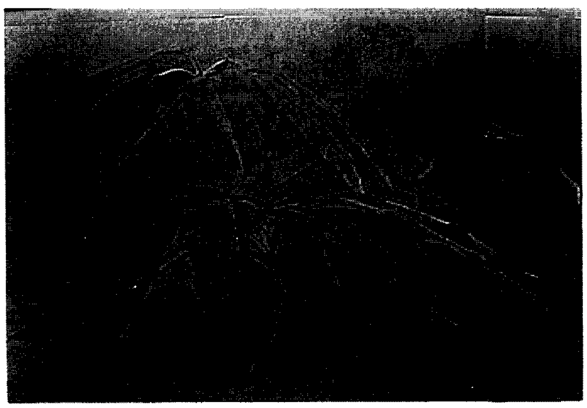
are computed from $\hat{f}$. Note that the phase of $\hat{p}$ at each point (ω_1, ω_2) is identical to that of $\hat{f}$, but the magnitude is one. Similarly, the respective magnitudes of $\hat{m}$ and $\hat{f}$ are the same at each point, but the phase of $\hat{m}$ is identically zero. The inverse DFT's of $\hat{m}$ and $\hat{p}$ are computed to obtain images with values $m(x, y)$ and $p(x, y)$, respectively. The symmetry necessary for m and p to be real valued is implied by f being real valued. The images are normalized to a predetermined energy and displayed.

Fig. 2(a)–(b) was obtained (as m and p, respectively) from the image in Fig. 1 by the procedure just described. The results obtained are similar to those in [5]; subjectively, at least, the phase-only image resembles the original image far more than does the magnitude-only image. These empirical results suggest that, in the case of normalized images, magnitude information may be completely lost and replaced by constant value and the image will still be recognizable provided the phase information is retained. On the other hand, if the phase information is lost and then replaced by a constant value, the image becomes unrecognizable even if the magnitude information is preserved.

$$\frac{\partial \mathcal{N}F(\boldsymbol{b}, \Theta)}{\partial b_{j,k}}(x,y) = \frac{e^{b_{j,k}} e^{i(jx+ky+\theta_{j,k})} \sum\sum e^{2b_{m,n}} - e^{2b_{j,k}} \sum\sum e^{b_{m,n}} e^{i(mx+ny+\theta_{m,n})}}{||f||^3}$$
$$= \frac{||f||^2 c_{j,k} e^{i(jx+ky+\theta_{j,k})} - c_{j,k}^2 \sum\sum c_{m,n} e^{i(mx+ny+\theta_{m,n})}}{||f||^3}$$



(a)



(b)

Fig. 2. (a) Magnitude-only image $m(x, y)$ obtained from the image f of Fig. 1; (b) phase-only image $p(x, y)$ obtained from f.

Magnitude Loss Distortion

Consider an image f on $\boldsymbol{I}^2$ that is bandlimited in the sense that its Fourier coefficients $a_{m,n}$ are zero for $|n| > N$ and $|m| > M$. For computational simplicity, it is convenient to assume $a_{m,n} \neq 0$ for all $|m| \leq M$ and $|n| \leq N$. As discussed above, the values of such an image may be written in terms of its phase and magnitude coefficients as

$$f(x, y) = \text{Re} \sum_{m=-M}^{M} \sum_{n=-N}^{N} c_{m,n} e^{i(mx+ny+\theta_{m,n})}$$

and the values of the corresponding normalized image $\mathcal{N}f$ may be expressed as

$$\mathcal{N}f(x, y) = \text{Re} \frac{\sum\sum c_{m,n} e^{i(mx+ny+\theta_{m,n})}}{(\sum\sum c_{m,n}^2)^{1/2}}. \quad (4)$$

Removing the magnitude information from the signal by setting the magnitude of each nonzero Fourier coefficient c_k to a constant value $\lambda > 0$ yields the signal p with values

$$p(x, y) = \text{Re} \sum\sum \lambda e^{i(mx+ny+\theta_{m,n})}.$$

The energy normalized image $\mathcal{N}p$ corresponding to p has values

$$\begin{aligned} \mathcal{N}p(x, y) &= \frac{\text{Re} \sum\sum \lambda e^{i(mx+ny+\theta_{m,n})}}{(\sum\sum \lambda^2)^{1/2}} \\ &= \frac{\text{Re} \sum\sum e^{i(mx+ny+\theta_{m,n})}}{\sqrt{K}} \end{aligned} \quad (5)$$

where $K \triangleq (2N + 1)(2M + 1)$ is the number of magnitude coefficients set equal to λ.

The quantitative effect of this loss of magnitude information may be evaluated in terms of the following:

Definition: The *mean-square distortion* to the image $f \in \boldsymbol{L}^2(\boldsymbol{I}^2)$ introduced by a transformation $\mathcal{T} : \boldsymbol{L}^2(\boldsymbol{I}^2) \to \boldsymbol{L}^2(\boldsymbol{I}^2)$ is defined for each $(x, y) \in \boldsymbol{I}^2$ by

$$D(x, y) \triangleq E|\mathcal{N}f(x, y) - \mathcal{N}\mathcal{T}f(x, y)|^2$$

where the expectation assumes the phase coefficients $\theta_{m,n}$ are independent random variables that are uniformly distributed on $\boldsymbol{I}$.

Theorem: The mean-square distortion to the image f introduced by the magnitude-loss transformation $f \mapsto p$ discussed above is

$$D_p(x, y) = 1 - \frac{\sum\sum c_{m,n}}{\sqrt{K}(\sum\sum c_{m,n}^2)^{1/2}} \quad (6)$$

for all $(x, y) \in \boldsymbol{I}^2$.

Proof: From (4) and (5)

$$\begin{aligned} &\mathcal{N}f(x, y) - \mathcal{N}p(x, y) \\ &= \sum_{|m|\leq M} \sum_{|n|\leq N} \left\{ \left[\frac{c_{m,n}}{(\sum\sum c_{j,k}^2)^{1/2}} - \frac{1}{\sqrt{K}} \right] \cos(mx + ny + \theta_{m,n}) \right\} \end{aligned}$$

so that

$$\begin{aligned} D_p(x, y) &= \frac{1}{2} \sum_m \sum_n \left[\frac{c_{m,n}}{(\sum\sum c_{j,k}^2)^{1/2}} - \frac{1}{\sqrt{K}} \right]^2 \\ &= 1 - \frac{\sum\sum c_{m,n}}{\sqrt{K}(\sum\sum c_{j,k}^2)^{1/2}} \end{aligned}$$

■

Geometry of Magnitude-Loss Distortion

Proceeding from the above theorem, it is possible to describe geometrically the region in $\mathbb{R}^K$ corresponding to vectors of Fourier coefficients $\boldsymbol{c} = (c_{-M,-N}, \ldots, c_{-M,N}, c_{-M+1,-N}, \ldots, c_{-M+1,N}, \ldots, c_{M,N})$ that yield the same value of D_p. It is also possible to introduce a probability measure in order to indicate how likely it is that significant distortions will be caused by setting all magnitude coefficients $c_{m,n}$ to a constant value.

Denote by $\boldsymbol{P}^K$ the region in $\mathbb{R}^K$ where all coordinates are positive. The subset of $\boldsymbol{P}^K$ on which $D_p = \delta$ is given by

$$1 - \frac{\sum\sum c_{m,n}}{\sqrt{K}(\sum\sum c_{j,k}^2)^{1/2}} = \delta$$

which is the equation of a right circular cone in $\mathbb{R}^N$ with vertex at the origin and axis $c_{-M,-N} = c_{-M,-N+1} = \cdots = c_{M,N}$. Note that the entire cone may not be contained in $\boldsymbol{P}^K$. In this case, the set of Fourier coefficients on which D_p is constant is the intersection of the cone with $\boldsymbol{P}^K$. The region $\boldsymbol{C}_\delta$ in which $D_p \leq \delta$ is the subset of $\boldsymbol{P}^K$ bounded by this cone, hence:

Theorem: If the magnitude coefficients $c_{m,n}$ have joint density $\phi(\boldsymbol{c})$, then

$$\Pr\{D_p \leq \delta\} = \int_{\boldsymbol{C}_\delta} \phi(\boldsymbol{c}) d\boldsymbol{c}.$$

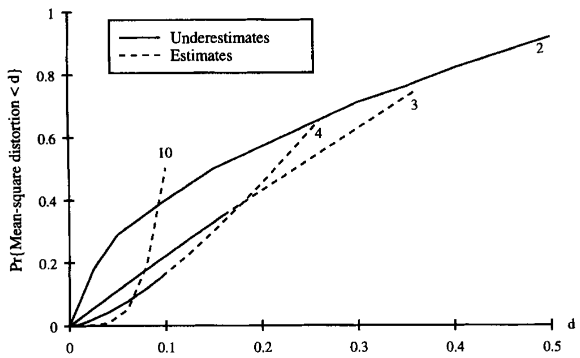


Fig. 3 Partial distribution function estimates for the relative mean-square error distortion D_p for $K = 2$, 3, 4, and 10 coefficients.

In particular, if the coefficients are concentrated near the axis of the cone, the mean square distortion will be small. If, for example, the sample variance of the coefficients is small compared with the square of their average

$$\frac{1}{K}\sum\sum c_{m,n}^2 - \frac{1}{K^2}\Big(\sum\sum c_{m,n}\Big)^2 \triangleq \epsilon \ll \Big(\frac{1}{K}\sum\sum c_{m,n}\Big)^2$$

then

$$D_p(x,y) \approx 1 - \frac{\sum\sum c_{m,n}}{\sum\sum c_{m,n} + K\epsilon} \ll 1.$$

Assume now that the vector $c/||c||$ is uniform on the intersection of the unit sphere S^{K-1} (in $\mathbb{R}^K$) and P^K. Denote $\Omega_\delta \triangleq S^{K-1} \cap P^K \cap C_\delta$. Then

$$\Pr\{D_P \le \delta\} = \frac{\text{Area}(\Omega_\delta)}{\text{Area}(S^{K-1} \cap P^K)}.$$

Call the angle of the cone Ψ_δ. For $\Psi_\delta < \cos^{-1}\left(\sqrt{\frac{K-1}{K}}\right)$, Ω_δ is a spherical cap and Area$(\Omega_\delta) >$ Area($(K-1)$-dimensional disk of radius $\sin(\Psi_\delta)$) so that

$$\frac{\text{Area}(\Omega_\delta)}{\text{Area}(S^{K-1} \cap P^K)} \ge \frac{2^K(\sin\Psi_\delta)^{K-1}\Gamma(\frac{K}{2}+1)}{K\sqrt{\pi}\Gamma(\frac{K-1}{2}+1)}.$$

The solid curves in Fig. 3 show partial distribution function underestimates for D_p corresponding to angles $\Psi_\delta \le \cos^{-1}\left(\sqrt{\frac{K-1}{K}}\right)$ that were generated using this expression. The broken curves are continuations of these estimates corresponding to angles $\Psi_\delta > \cos^{-1}\left(\sqrt{\frac{K-1}{K}}\right)$ for which the probability that $D_p \le \delta$ may be overestimated.

Phase Loss Distortion

Consider now the image m which is formed by setting all phase angles $\theta_{m,n}$ in the original image f to zero. Then

$$D_m(x,y) \triangleq E|\mathcal{N}f(x,y) - \mathcal{N}m(x,y)|^2 \ge \frac{1}{2}.$$

Thus, under the statistical assumptions on the phase angles, the distortion to a normalized image caused by loss of phase information is significant regardless of the distribution of its magnitude coefficients.

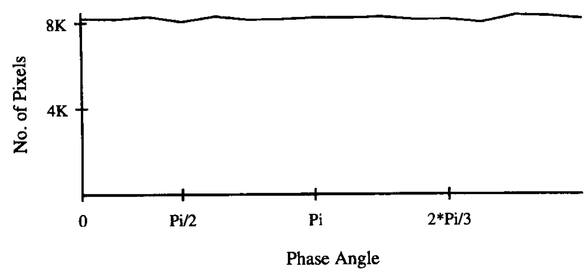


Fig. 4. Phase histogram showing the distribution of the phase angles in the spectrum of the image of Fig. 1.

V. Discussion and Conclusion

The interaction of nonlinear mechanisms for energy and dynamic range regulation with other common components of signal and image processing systems has received little attention in the literature. One possible reason for this is the deceptive simplicity of important nonlinear mechanisms, such as energy normalization. When a signal f is normalized to unit energy, the effect is that its values are all multiplied by a single positive normalization constant, $||f||^{-1}$. Thus it is tempting to regard normalization of a signal as a simple multiplication by a constant, thereby creating the impression that it should commute with linear signal processing operations such as spectrum analysis. It is the dependence of the normalization constant on the signal that makes the operation nonlinear and complicates its interaction with linear operations.

When normalization is regarded as an operator $\mathcal{N}$ on a normed signal space (*e.g.*, $L^2(I^2)$), its consequences are more apparent. From this perspective, it is evident that $\mathcal{N}$ is a nonlinear, many-to-one operator that maps each nonzero signal in the space to a signal of unit norm, effectively collapsing the space onto its unit sphere. This means that $||\mathcal{N}(f) - \mathcal{N}(g)||$ may be less than, greater than, or equal to $||f - g||$; normalization significantly changes which signals are "close together" in the space.

This correspondence shows that normalization interacts with Fourier decomposition in such a way as to accentuate the importance of phase information relative to magnitude information. The underlying assumption that the phase angles are independent and uniformly distributed is clearly an approximation. Histograms of phase coefficients, as depicted in Fig. 4, and statistical significance tests suggest that the uniform distribution is a reasonably good model for the phase coefficients of typical pictures. The independence assumption is more difficult to test. Phase angles along edges and texture boundaries, for example, will certainly be correlated while those from distinct regions of typical images will not. The role of this model in the above results is the subject of ongoing research.

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Optimal Edge-Preserving Hybrid Filters

Xin Wang

***Abstract*— Based on the least mean square error criterion at edge position, a new algorithm is presented to obtain a class of optimal edge-preserving hybrid filters that include the Lee additive filter and the gradient inverse weighted filter. As a result, we show that the gain of the Lee additive filter is not optimal and derive the optimal form of this filter. In addition, some examples are given in this correspondence.**

I. Introduction

In image processing, one of the important tasks is to suppress noise. Linear techniques posses mathematical simplicity but have the disadvantage of blurring edges. Thus, a number of nonlinear filters were presented. The median filter [1] is one of the well-known nonlinear filters. Unfortunately, although the median filter can preserve edges, it is not good enough at smoothing Gaussian white noise.

Consequently, some hybrid filters were developed. The generalized median filter [2] and the FIR-median hybrid filter [3] are some examples. Those edge-preserving nonlinear filters may give better results than the median filter in smoothing Gaussian white noise.

Recently, the research interests of this field were centered on finding some optimal nonlinear filters that can give the best results in signal processing [4].

Without doubt, using the least mean square error (LMSE) criterion, we can derive some optimal filters in smoothing Gaussian white noise, but in general cases, they may lose the property of preserving edges.

In this correspondence, using the LMSE criterion at edge positions, we present a new algorithm to obtain a class of hybrid filters that can give the best results both in preserving edges and in smoothing Gaussian white noise. In addition, the optimal gain of the Lee additive filter [5], [6] is derived.

II. The Optimal Edge-Preserving Weights

Let the original image $x(i, j)$ be a random field with mean μ and variance σ_0^2. Then, the noisy image can be represented by

$$y(i,j) = x(i,j) + n(i,j) \tag{1}$$

where $n(i,j)$ is signal-independent Gaussian white noise with mean zero and variance σ_n^2.

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Suppose we have p unbiased estimators of the image given by

$$\hat{x}_r(i,j) = x(i,j) + v_r(i,j) \qquad (r = 1,\ldots,p) \tag{2}$$

where $v_1(i,j),\ldots,v_p(i,j)$ are the estimate errors. For simplicity, we assume they are independent noise with mean zero and variance $\sigma_1^2,\ldots,\sigma_p^2$, respectively.

Our purpose is to determine a new estimate $\hat{x}(i,j)$, which has the form

$$\hat{x}(i,j) = \sum_{r=1}^{p} K_r \hat{x}_r(i,j) \tag{3}$$

where K_r are the weights. Obviously, to guarantee that the $\hat{x}(i,j)$ is the unbiased estimate, we have

$$\sum_{r=1}^{p} K_r = 1. \tag{4}$$

Thus, (3) becomes

$$\hat{x}(i,j) = \sum_{r=1}^{p-1} K_r[\hat{x}_r(i,j) - \hat{x}_p(i,j)] + \hat{x}_p(i,j). \tag{5}$$

It is known that the LMSE estimation requires

$$\frac{\delta\{E[x(i,j) - \hat{x}(i,j)]^2\}}{\delta K_t} = 0. \quad (t = 1,\ldots,p-1) \tag{6}$$

From (2), (4), (5), and (6), the optimal weights are given by

$$K_t = (1/\sigma_t^2) \Bigg/ \left(\sum_{r=1}^{p} 1/\sigma_r^2\right). \qquad (t = 1,\ldots,p) \tag{7}$$

We call them optimal noise-smoothing weights.

Unfortunately, if some of the estimators used in (3) produce a large error at edge positions, based on those weights, the edges may be blurred. Thus, it is necessary to obtain some new weights that can give better results in preserving edges.

Assume that at the edge position, the p estimators can be rewritten as

$$\hat{x}_r(i,j) = a_r x(i,j) + v_r(i,j) \qquad (r = 1,\ldots,p) \tag{8}$$

where $a_r \quad (r = 1,..,p)$ are constants that represent the ability to preserve edges of the estimators $\hat{x}_r(i,j)$. Then, from (3), (4), (6), and (8), we have

$$(1-a_p)(a_t-a_p)\sigma_0^2 - \sum_{r=1}^{p-1} K_r[(a_r-a_p)(a_t-a_p)\sigma_0^2 + \sigma_p^2] - K_t\sigma_t^2 + \sigma_p^2 = 0 \tag{9}$$

where $t = 1,\ldots,p-1$.

Therefore, from (4) and (9), the weights $K_r (r = 1,\ldots,p)$ can be evaluated easily. We call them optimal edge-preserving weights.

In this correspondence, we only consider that there are two estimators $y_1(i,j)$ and $y_2(i,j)$ in (3), and let

$$y_1(i,j) = y(i,j) = x(i,j) + n(i,j) \tag{10}$$
$$y_2(i,j) = bx(i,j) + w(i,j). \tag{11}$$

Then, (3) becomes

$$\hat{x}(i,j) = Ky(i,j) + (1-K)y_2(i,j) \tag{12}$$