Construction of Discrete Orthogonal Wavelet Bases

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Abstract – *In general, two sequences formed by uniformly sampling two orthogonal signals will not be orthogonal. This paper presents families of discrete orthonormal wavelet bases for ℓ^2 that are obtained by sampling of certain dyadic orthonormal wavelet bases of L^2 over a bounded frequency band.*

1. INTRODUCTION

Numerous wavelet bases for $L^2(\mathbb{R})$ have been described in recent mathematical and engineering literature. Because of their tractability in applications, dyadic orthonormal wavelet bases have received considerable attention. Use of such bases in discrete-time settings generally involves sampling of the mother wavelet at uniform intervals which are power-of-two multiples of a fixed interval T. This raises the issue that the sample sequences obtained from orthogonal time-scale replicates of the mother wavelet may not be orthogonal.

This paper investigates the problem of obtaining (discrete-time) dyadic wavelet bases for ℓ^2 from (continuous-time) dyadic wavelet bases of L^2. In particular, a construction of S. Mallat is extended and the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem is applied to obtain a family of discrete orthonormal wavelet bases for ℓ^2.

2. BASIS CONSTRUCTION

In the frequency domain, define

$$\hat{W}^a(\omega) = \begin{cases} 1 & \text{if } \pi < |\omega| \le 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, define $\hat{W}^a_{m,n}(\omega) = 2^{m/2}\hat{W}^a(2^m\omega)e^{-i2^m n\omega}$ or $W^a_{m,n}(t) = 2^{-m/2}W^a(t/2^m - n)$. Then $\hat{W}^a_{m,n}$ with $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ form an orthogonal basis of $L^2[-\pi,\pi]$. Moreover, each $W^a_{m,n}$ is bandlimited to $[-\pi,\pi]$ and hence by the WKS theorem may be represented by samples $\{W^a_{m,n}(k)\}_{k\in\mathbb{Z}}$ or $\{W_{m,n}(k)\}_{k\in\mathbb{Z}}$.

Theorem 1: The sample sequences $\{W_{m,n}(k)\}_{k\in\mathbb{Z}}$ are orthonormal; i.e., for arbitrary dyadic dilation indices $m \in \mathbb{N}$ and $m' \in \mathbb{N}$ and integer time shifts $n \in \mathbb{Z}$ and $n' \in \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} W_{m,n}(k)W^*_{m',n'}(k) = \begin{cases} 0 & \text{if } (m,n) \ne (m',n') \\ 1 & \text{if } (m,n) = (m',n') \end{cases} \tag{2}$$

Proof: From above, $\{W^a_{m,n}\}$ is an orthogonal basis of L^2 and $\{W_{m,n}\}$ is generated by sampling the corresponding analog function with sampling interval $T = 1$. Note that dyadic dilation index m is restricted to be a natural number and the time shift index n to be an integer.

Let $\hat{W}_{m,n}$ be the discrete time Fourier transform (DTFT) of the discrete signal $W_{m,n}$

$$\hat{W}_{m,n}(\omega) = \sum_k W_{m,n}(k)e^{-i\omega k}$$

The relationship between $W^a_{m,n}$ and $W_{m,n}$ is

$$\hat{W}_{m,n}(\omega) = \sum_k \hat{W}^a_{m,n}(\omega - 2\pi k) \tag{3}$$

Since $\hat{W}^a_{m,n}(\omega) = 0$ when $|\omega| > \pi$,

$$\hat{W}_{m,n}(\omega) = \hat{W}^a_{m,n}(\omega) \text{ when } |\omega| \le \pi \tag{4}$$

By Parseval's theorem

$$\sum_k W_{m,n}(k)\, W^*_{m',n'}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{W}_{m,n}(\omega)\hat{W}^*_{m',n'}(\omega)\, d\omega$$

From equation (4),

$$\begin{aligned} \langle W_{m,n}, W_{m',n'}\rangle &= \sum_k W_{m,n}(k)W^*_{m',n'}(k) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{W}^a_{m,n}(\omega)\hat{W}^{a*}_{m',n'}(\omega)\, d\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{W}^a_{m,n}(\omega)\hat{W}^{a*}_{m',n'}(\omega)\, d\omega \\ &= \langle W^a_{m,n}, W^{a*}_{m',n'}\rangle \end{aligned}$$

Orthonormality of the analog functions $\{W^a_{m,n}\}$ thus implies orthonormality of the sample sequences $\{W_{m,n}\}$. ∎

Theorem 2: The sequences $\{W_{m,n}(k)\}_{k\in\mathbb{Z}}$ span ℓ^2. Thus they comprise an orthonormal basis for the space of discrete-time finite energy signals.

Proof: For any discrete signal $f \in \ell^2$ whose DTFT $\hat{f}$ is periodic with period 2π, an analog signal f^a can be constructed by inverse Fourier transform of the frequency domain function defined by

$$\hat{f}^a(\omega) = \begin{cases} \hat{f}(\omega) & |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

The function f^a can be expressed as a weighted sum of the orthogonal basis elements $W^a_{m,n}$. Thus

$$f^a(t) = \sum_m \sum_n a_{m,n} W^a_{m,n}(t)$$

and

$$f(k) = \sum_m \sum_n a_{m,n} W_{m,n}(k) \qquad \forall k \in \mathbb{Z} \tag{5}$$

Hence $\{W_{m,n}\}$ spans ℓ^2. ∎

The procedure just described may be applied to other frequency-domain L^2 wavelet basis constructions using bandlimited wavelets to yield other discrete orthogonal bases for ℓ^2.

Some additional properties of the above construction are listed below without proof. Let $f(k)$ be as in equation (5) and define

$$g(k) = \sum_m \sum_n b_{m,n} W_{m,n}(k) \quad \text{and} \quad h(k) = \sum_m \sum_n c_{m,n} W_{m,n}(k)$$

Then

- *Linearity:* If $h = f + g$ then $c_{m,n} = a_{m,n} + b_{m,n}$
- *Convolution:* If $h = f * g$ (i.e., $h(n) = \sum_k f(k)g(n-k)$) then $c_{m,n} = a_{m,n} * b_{m,n} = 2^{m/2}\sum_k a_{m,n-k}\cdot b_{m,k}$
- *Dyadic dilation:* $W_{m,0}(k) = \sqrt{2}W_{m+1,0}(2k)$

Reference

[1] S. G. Mallat, *Multiresolution Representations and Wavelets.* Ph.D. thesis, University of Pennsylvania, August 1988.