Construction of Discrete Orthogonal Wavelet Bases

Chao Wei and Douglas Cochran
Telecommunications Research Center
Arizona State University
Tempe, Arizona 85287-7206

Abstract — In general, two sequences formed by uniformly sampling two orthogonal signals will not be orthogonal. This paper presents families of discrete orthonormal wavelet bases for \( L^2 \) that are obtained by sampling of certain dyadic orthonormal wavelet bases of \( L^2 \) over a bounded frequency band.

1. INTRODUCTION

Numerous wavelet bases for \( L^2(\mathbb{R}) \) have been described in recent mathematical and engineering literature. Because of their tractability in applications, dyadic orthonormal wavelet bases have received considerable attention. Use of such bases in discrete-time settings generally involves sampling of the mother wavelet at uniform intervals which are power-of-two multiples of a fixed interval. This raises the issue that the sample sequences obtained from orthogonal time-scale replicates of the mother wavelet may not be orthogonal.

This paper investigates the problem of obtaining (discrete-time) dyadic wavelet bases for \( L^2 \) from (continuous-time) dyadic wavelet bases of \( L^2 \). In particular, a construction of S. Mallat is extended and the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem is applied to obtain a family of discrete orthonormal wavelet bases for \( L^2 \).

2. BASIS CONSTRUCTION

In the frequency domain, define

\[
\tilde{W}^a(\omega) = \begin{cases} 
1 & \text{if } |\omega| \leq 2\pi \\
0 & \text{otherwise}
\end{cases}
\]

and for \( m \in \mathbb{N} \) and \( n \in \mathbb{Z} \), define \( \tilde{W}^a_m(n, \omega) = 2^{-m/2} \tilde{W}^a(2^m \omega) e^{-i 2^m \omega n} \) or \( W^a_m(t) = 2^{-m/2} \tilde{W}^a(2^m \omega) e^{-i 2^m \omega t} \). Then \( \tilde{W}^a_m(n) \) with \( m \in \mathbb{N} \) and \( n \in \mathbb{Z} \) form an orthogonal basis of \( L^2[0, 1] \). Moreover, each \( \tilde{W}^a_m(n) \) is bandlimited to \([0, 1]\) and hence by the WKS theorem may be represented by samples \( \{\tilde{W}^a_m(n)\}_{n \in \mathbb{Z}} \) or \( \{\tilde{W}^a_m(n)\}_{n \in \mathbb{Z}} \).

Theorem 1: The sample sequences \( \{\tilde{W}^a_m(n)\}_{n \in \mathbb{Z}} \) are orthonormal; i.e., for arbitrary dyadic dilation indices \( m \in \mathbb{N} \) and \( m', n' \in \mathbb{N} \) and integer time shifts \( n \in \mathbb{Z} \) and \( n' \in \mathbb{Z} \),

\[
\sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{Z}} W^a_{m,n}(k) W^a_{m',n'}(k) = \begin{cases} 
1 & \text{if } (m, n) \neq (m', n') \\
0 & \text{if } (m, n) = (m', n')
\end{cases}
\]

Proof: From above, \( \{\tilde{W}^a_m(n)\}_{n \in \mathbb{Z}} \) is an orthogonal basis of \( L^2 \) and \( \{\tilde{W}^a_m(n)\}_{n \in \mathbb{Z}} \) is generated by sampling the corresponding analog function with sampling interval \( T = 1 \). Note that dyadic dilation index \( m \) is restricted to be a natural number and the time shift index \( n \) to be an integer.

Let \( \hat{W}^a_m(n) \) be the discrete time Fourier transform (DTFT) of the discrete signal \( W^a_m(n) \)

\[
\hat{W}^a_m(n) = \sum_k W^a_m(n,k) e^{-i 2\pi k n}
\]

The relationship between \( W^a_m(n) \) and \( W^a_m(n) \) is

\[
W^a_m(n) = \sum_k \hat{W}^a_m(n) e^{-i 2\pi k n}
\]

Since \( \tilde{W}^a_m(n) = 0 \) when \( |\omega| > \pi \),

\[
\hat{W}^a_m(n) = \hat{W}^a_m(n) \text{ when } |\omega| \leq \pi
\]

By Parseval's theorem

\[
\sum_k W^a_{m,n}(k) W^a_{m,n}^*(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{W}^a_{m,n}(\omega) \hat{W}^a_{m,n}^*(\omega) d\omega
\]

From equation (4),

\[
(W^a_m(n), W^a_{m',n'}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{W}^a_{m,n}(\omega) \hat{W}^a_{m',n'}^*(\omega) d\omega
\]

Orthogonality of the analog functions \( \{W^a_m(n)\}_{n \in \mathbb{Z}} \) thus implies orthonormality of the sample sequences \( \{W^a_m(n)\}_{n \in \mathbb{Z}} \).

Theorem 2: The sequences \( \{W^a_m(n)\}_{n \in \mathbb{Z}} \) span \( L^2 \). Thus they comprise an orthonormal basis for the space of discrete-time finite energy signals.

Proof: For any discrete signal \( f \in L^2 \) whose DTFT \( \hat{f} \) is periodic with period \( 2\pi \), an analog signal \( \tilde{f} \) can be constructed by inverse Fourier transform of the frequency domain function defined by

\[
\tilde{f}(\omega) = \begin{cases} 
\hat{f}(\omega) & |\omega| \leq \pi \\
0 & \text{otherwise}
\end{cases}
\]

The function \( \tilde{f} \) can be expressed as a weighted sum of the orthogonal basis elements \( \tilde{W}^a_m(n) \) . Thus

\[
f(t) = \sum_m \sum_n a^m_n \tilde{W}^a_m(n)\]

and

\[
a^m_n = \frac{\int_{-\pi}^{\pi} f(t) \tilde{W}^a_m(n,t) dt}{\int_{-\pi}^{\pi} \tilde{W}^a_m(n,t) \tilde{W}^a_m(n,t) dt}
\]

Hence \( \{W^a_m(n)\}_{n \in \mathbb{Z}} \) spans \( L^2 \).

The procedure just described may be applied to other frequency-domain \( L^2 \) wavelet basis constructions using bandlimited wavelets to yield other discrete orthogonal bases for \( L^2 \).

Some additional properties of the above construction are listed below without proof.

1. Linearity: If \( h = f + g \) then \( c^a_m(n) = a^m_n + b^m_n \).
2. Convolution: If \( h = f * g \) (i.e., \( h(n) = \sum_k f(k)g(n-k) \)) then \( c^a_m(n) = a^m_n * b^m_n = 2^{-m/2} \sum_k a^m_{n-k} b^m_k \).

3. Dyadic dilation: \( W^a_m(n) = \sqrt{2} \tilde{W}^a_{m+1,0}(2n) \)

Reference