INVARIANCE OF THE GENERALIZED COHERENCE ESTIMATE WITH RESPECT TO REFERENCE CHANNEL STATISTICS

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Abstract— **The probability distribution function of the three-channel generalized coherence estimate is shown not to depend on the statistical behavior of the data on one channel provided the other two channels contain white gaussian noise and all channels are independent. A technique for multiple-channel matched filtering that is made viable by this invariance result is also discussed.**

I. Introduction

The ability to determine whether a common signal is present on two or more noisy channels is desirable for a variety of applications, particularly in situations where a signal source is to be detected and localized from measurements made at several sensors. In the case of two channels, a popular approach is to compare the magnitude-squared coherence (MSC) estimate computed using sample sequences from the two channels to a threshold corresponding to a predetermined probability of false alarm. Calculation of such detection thresholds depends upon knowledge of the probability distribution function (PDF) of the MSC estimate under suitable H_0 assumptions.

In [1], the PDF of the MSC estimate was derived assuming the two channels contain independent white gaussian sequences. In practice, however, one of the channels used in the MSC detection process is often known not to contain white noise. In a passive sonar setting, for example, a channel might be determined to contain a signal by a single-channel detector and then correlated against other channels with the hope of obtaining a two-channel detection and thereby deducing time difference of arrival (TDOA) information that will be useful in locating the source of the signal. For an active detection scenario, it is useful to establish detection thresholds that are valid when one channel contains a noise-free replica of the transmitted signal (matched filter).

This problem was addressed in [2], where the PDF of the MSC estimate was shown not to depend on the statistics of one of the two data sequences provided that the other sequence is white gaussian noise and the two sequences are independent. The H_0 assumptions were weakened further in [3], where a geometric argument was used to show that spherically symmetric distribution of the noise could replace the stronger gaussian assumption used in [2].

The generalized coherence (GC) estimate was introduced in [4] and its utility as a statistic for detecting the presence of a common signal on $M \geq 2$ noisy channels was studied in [5]. In particular, the PDF of the GC estimate was computed under the assumption that the sequences obtained from the M channels are independent white gaussian noise sequences. Detection thresholds for the three-channel GC estimate corresponding to a range of probabilities of false alarm are also given in this reference.

The results in the following section of this paper show that the PDF of the three-channel GC estimate given in [5] remains valid under H_0 assumptions that allow an arbitrary distribution of the data on one of the three channels provided it is independent from data on the other two channels. Such invariance suggests the utility of the GC estimate as a test statistic for detecting the presence of a weak signal on channels 2 and 3 when a strong signal is present on channel 1. This observation is discussed further in section IV, where the use of the GC estimate as the basis for an approach to multiple-channel matched filtering is considered.

II. The Generalized Coherence Estimate

Given M complex data sequences

$$\begin{aligned} x_1 &= (x_{1,1}, ..., x_{1,N})^T \\ &\vdots \\ x_M &= (x_{M,1}, ..., x_{M,N})^T \end{aligned}$$

the GC estimate formed from them is defined in [4] as

$$\gamma^2_{M,N}(x_1, ..., x_M) \triangleq 1 - \frac{g(x_1, ..., x_M)}{||x_1||^2 \cdots ||x_M||^2}$$

where $g(x_1, ..., x_M)$ denotes the determinant of the $M \times M$ Gram matrix

$$G(x_1, ..., x_M) \triangleq \begin{bmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_M \rangle \\ \vdots & & \vdots \\ \langle x_M, x_1 \rangle & \cdots & \langle x_M, x_M \rangle \end{bmatrix}$$

In these expressions, x^* denotes the complex conjugate of x,

$$\langle x_i, x_j \rangle \triangleq \sum_{n=1}^{N} x_{i,n} x^*_{j,n} \tag{1}$$

is the inner product of x_i and x_j, and

$$||x_i||^2 \triangleq \langle x_i, x_i \rangle \tag{2}$$

is the squared magnitude of x_i.

For $M = 2$, the GC estimate reduces to the familiar MSC estimate

$$\gamma^2_{2,N}(x_1, x_2) = \gamma^2(x_1, x_2) = \frac{|\langle x_1, x_2 \rangle|^2}{||x_1||^2 ||x_2||^2}$$

For $M = 3$, the case of particular interest in this paper, the GC estimate can be decomposed as follows:

$$\begin{aligned} \gamma^2_{3,N}(x_1, ..., x_M) &\triangleq 1 - \left(\frac{1}{||x_1||^2 ||x_2||^2 ||x_3||^2} \right) \\ &\cdot \det \begin{bmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \langle x_1, x_3 \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \langle x_2, x_3 \rangle \\ \langle x_3, x_1 \rangle & \langle x_3, x_2 \rangle & \langle x_3, x_3 \rangle \end{bmatrix} \\ &= \gamma^2_{2,N}(x_1, x_2) + \gamma^2_{2,N}(x_1, x_3) + \gamma^2_{2,N}(x_2, x_3) \\ &\quad - 2 \operatorname{Re} \frac{\langle x_1, x_2 \rangle \langle x_2, x_3 \rangle \langle x_1, x_3 \rangle^*}{||x_1||^2 ||x_2||^2 ||x_3||^2} \end{aligned} \tag{3}$$

Thus the three-channel GC estimate is the sum of the MSC estimates obtained from the three channel pairs minus a term incorporating data from all three channels.

III. Invariance Results

The distribution of the MSC estimate was shown in [3] not to depend on the behavior of x_1 provided that x_2 has a spherically symmetric distribution and is independent of x_1. In particular, if x_2 is a complex white gaussian process, then it is spherically symmetric and the MSC estimate will be invariant with respect to x_1 as long as it is independent of x_2. The goal of this section is to show that the distribution of the three-channel GC estimate does not depend on the behavior of one of the three sequences, provided:

1. The other two sequences are white gaussian noise; and
2. The three sequences are independent.

This result is based on the following lemma:

Lemma: Let a be an N-dimensional vector of independent, identically distributed complex gaussian random variables, each having independent real and imaginary parts. If b is an arbitrary N-dimensional complex random vector that is independent of a, then the distribution of the normalized inner product

$$\frac{\langle a, b \rangle}{||a|| \, ||b||} \tag{4}$$

does not depend on the distribution of b.

Proof (sketch): Denote $a \triangleq (x_1 + iy_1, ..., x_N + iy_N)^T$ and $b \triangleq (u_1 + iv_1, ..., u_N + iv_N)^T$. Define unit vectors in $\mathbf{R}^{2N}$ by

$$\begin{aligned} \alpha &\triangleq (x_1, x_2, ..., x_N, y_1, ..., y_N)^T / ||a|| \\ \beta_1 &\triangleq (u_1, u_2, ..., u_N, v_1, ..., v_N)^T / ||b|| \\ \beta_2 &\triangleq (-v_1, -v_2, ..., -v_N, u_1, ..., u_N)^T / ||b|| \end{aligned}$$

The normalized inner product (4) of the complex vectors a and b can be expressed in terms of the inner products of the real unit vectors α, β_1, and β_2 by

$$\frac{\langle a, b \rangle}{||a|| \, ||b||} = \langle \alpha, \beta_1 \rangle + i \langle \alpha, \beta_2 \rangle$$

By construction, the statistics of α depend only on those of a and both β_1 and β_2 depend only on b. Also, if a is as postulated, then α is uniformly distributed on the unit sphere in $\mathbf{R}^{2N}$. The inner product $\langle \alpha, \beta_1 \rangle$ represents the orthogonal projection of the real $2N$-dimensional unit vector α into the one-dimensional subspace B_1 of $\mathbf{R}^{2N}$ spanned by the unit vector β_1. Because α is uniformly distributed on the unit sphere in $\mathbf{R}^{2N}$, $\Pr \{ \langle \alpha, \beta_1 \rangle \leq r \}$ for $-1 \leq r \leq 1$ is equal to the fraction of the area of the unit sphere in $\mathbf{R}^{2N}$ that projects orthogonally into the interval $[-1, r]$ in B_1. Since α is independent of β_1, this projection does not depend on the orientation of β_1. Thus $\langle \alpha, \beta_1 \rangle$ does not depend on β_1 and hence the real part of (4) does not depend on b. The same argument shows that the imaginary part of (4) does not depend on b. Therefore the normalized inner product

$$\frac{\langle a, b \rangle}{||a|| \, ||b||}$$

does not depend on b, as claimed. ∎

In expression (3) for the three-channel GC estimate, independence of the sequences implies that $\gamma^2(x_2, x_3)$ does not depend on x_1. The additional assumption that x_2 and x_3 are white gaussian sequences implies that they have spherically symmetric distributions. Thus the results of [3] imply that the distributions of the first three terms in this sum do not depend on x_1 and the independence assumption implies that all three of these MSC estimates are independent random variables. By the lemma, none of the factors in the remaining term depend on x_1, and $\gamma^2_{3,N}(x_1, x_2, x_3)$ is invariant with respect to x_1.

IV. Multiple-Channel Matched Filtering

If x_1 is a noise-free signal sequence and one wishes to determine whether the sequences x_2 and x_3 contain x_1 corrupted by additive spherically symmetric noise, the above results suggest that the GC estimate $\gamma^2(x_1, x_2, x_3)$ might be used as a detection statistic. The H_0 PDF given in [4] is valid if x_2 and x_3 are white gaussian sequences and x_1, x_2, and x_3 are independent. Thus, with these assumptions on the "signal absent" situation, detection thresholds corresponding to desired false alarm rates can be established even though x_1 is a matched filter.

The signal absent hypotheses just discussed are not necessarily the correct ones for many multiple-channel detection applications. In the three-channel case, a sufficiently high SNR on *either* x_2 or x_3 would yield a detection. This is illustrated by the solid curve in Figure 1 which shows the signal-to-noise ratios (SNRs) on x_2 and x_3 necessary to obtain a three-channel detection when x_1 is a matched filter. If the SNR on x_2 is above about -8 dB in this scenario, for example, the GC detector will decide on H_1 even if x_3 contains only noise.

This problem may be addressed by computing the MSC estimates $\gamma^2(x_1, x_2)$ and $\gamma^2(x_1, x_3)$ in addition to the three-channel GC estimate. A detection algorithm using all three computed values can be implemented as follows:

1. If $\gamma^2_{3,N}(x_1, x_2, x_3)$ is below the three-channel detection threshold, decide the signal is not present on channels 2 and 3;
2. If $\gamma^2_{3,N}(x_1, x_2, x_3)$ is above the three-channel detection threshold and *neither* of the MSC estimates $\gamma^2(x_1, x_2)$ or $\gamma^2(x_1, x_3)$ are above the two-channel detection threshold, decide the signal is present on channels 2 and 3;
3. If $\gamma^2_{3,N}(x_1, x_2, x_3)$ is above the three-channel detection threshold and *both* of the MSC estimates $\gamma^2(x_1, x_2)$ and $\gamma^2(x_1, x_3)$ are above the two-channel detection threshold, decide the signal is present on channels 2 and 3;
4. If $\gamma^2_{3,N}(x_1, x_2, x_3)$ is above the three-channel detection threshold and $\gamma^2(x_1, x_2)$ is above the two-channel detection threshold but $\gamma^2(x_1, x_3)$ is below the two-channel detection threshold, decide the signal is present on channel 2 only; and
5. If $\gamma^2_{3,N}(x_1, x_2, x_3)$ is above the three-channel detection threshold and $\gamma^2(x_1, x_3)$ is above the two-channel detection threshold but $\gamma^2(x_1, x_2)$ is below the two-channel detection threshold, decide the signal is present on channel 3 only.

Examination of Figure 1 reveals that there will be cases in which the GC estimate yields a three-channel detection while neither x_2 nor x_3 has sufficient SNR to yield a two-channel MSC detection. This characteristic has been noted in several simulation results, and is more pronounced in some cases than in the case of Figure 1.

V. Discussion and Conclusions

Previous work involving the use of the GC estimate as a detection statistic has focused on its utility in passive surveillance applications. Invariance of the GC estimate to reference channel statistics extends its utility to active situations by allowing one channel to contain a noise-free signal replica without altering the detection thresholds established for use in passive detection.

Although invariance to reference channel statistics is shown in this short paper only for the three-channel GC estimate, preliminary results indicate that it holds for larger numbers of channels as well. The authors are seeking a proof based more directly on an interpretation of the GC estimate as a volume in $\mathbf{C}^N$.

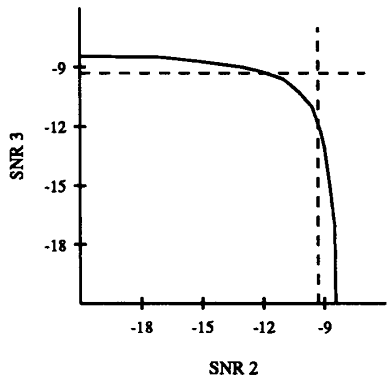

Figure 1: Matched filter performance for a white signal in white noise with $N = 512$, $P_f = 0.001$, and $P_d = 0.5$ (from simulations). The solid curve is the locus of SNRs on channels 2 and 3 necessary to achieve a detection with a three-channel GC detector having infinite SNR on channel 1. Note that a strong signal on channel 2 will result in a detection even if channel 3 contains only noise. The dashed lines indicate the SNRs necessary to achieve a two-channel matched filter detection using a MSC detector.

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VII. References

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