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LINEAR FILTERING ON SIGNALS REPRESENTED BY NON-UNIFORMLY SPACED SAMPLES

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Abstract— **Earlier work has established that signals formed by composition of a bandlimited signal with a bijective *time warping* function can be reconstructed from prescribed sequences of non-uniformly spaced samples. This paper describes a technique for linear filtering of this class of signals directly from sequences of non-uniformly spaced samples. An example of low-pass filtering a non-bandlimited signal represented by non-uniformly spaced samples using this method is presented.**

I. Introduction

In [1], Clark, Palmer, and Lawrence presented a coordinate transformation method that allows the reconstruction of a class of signals formed by time warping of bandlimited signals from prescribed sequences of non-uniformly spaced sample values. Subsequently, Clark, Cochran, and Djebeniani [2],[3] have studied the space of signals to which the technique applies. This paper presents a technique for linear filtering of this class of signals directly from sequences of non-uniformly spaced samples.

Let B denote the set of bandlimited signals $f : \mathbf{R} \to \mathbf{C}$ and let Γ denote the set of bijective functions $\gamma : \mathbf{R} \to \mathbf{R}$. If f is a signal in B with bandwidth $\omega_0 = \frac{\pi}{T}$ and is sampled at the points $t_n = nT$, the Shannon sampling theorem allows the exact reconstruction of f from its uniformly spaced samples as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{f(nT)\sin(\omega_0(t-nT))}{\omega_0(t-nT)} \qquad (1)$$

Let h be a non-uniformly sampled signal with the nth sample time t_n not necessarily equal to nT. As was shown in [3], if one can find a signal $f \in B$ with bandwidth ω_0 and a function $\gamma \in \Gamma$ with $\gamma(t_n) = nT$ such that $h(t) = f(\gamma(t))$, then h can be exactly reconstructed as follows:

$$h(t) = f(\gamma(t)) = \sum_{n=-\infty}^{\infty} \frac{h(t_n)\sin(\omega_0(\gamma(t)-nT))}{\omega_0(\gamma(t)-nT)}$$

$$= \sum_{n=-\infty}^{\infty} h_n \operatorname{sinc}(\omega_0(\gamma(t)-nT)) \qquad (2)$$

where $h_n = h(\gamma^{-1}(nT))$ and $\operatorname{sinc}(x) = \sin(x)/x$. Given the sample values h_n, linear filtering of the signal h may be accomplished by reconstructing it explicitly using equation (2), and then applying the filter. The following section describes an approach in which it is not necessary to reconstruct h. The filtering may be performed directly on the non-uniform sample sequence h_n. Implementation of a filter using this approach requires knowledge of γ and the bandwidth of f, but no other *a priori* knowledge about f. Thus much of the computation involved in implementing a filter in this way may be done before the sample sequence is obtained.

II. Theoretical Analysis

Let $h = f \circ \gamma$ be a time warped bandlimited signal represented by the sequence of non-uniform samples h_n. Suppose h is being input to a linear time invariant filter with impulse response φ. The output signal y can be found by convolving h with φ, which is expressed as

$$y(t) = (h * \varphi)(t) = \int_{-\infty}^{\infty} h(s)\varphi(t-s)ds \qquad (3)$$

for $-\infty < t < \infty$. Substituting (2) in (3) yields

$$y(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h_n \operatorname{sinc}(\omega_0(\gamma(s)-nT))\varphi(t-s)ds$$

Assuming the necessary convergence conditions and taking the summation outside the integral gives

$$y(t) = \sum_{n=-\infty}^{\infty} h_n \int_{-\infty}^{\infty} \mathrm{sinc}(\omega_0(\gamma(s) - nT))\varphi(t-s)ds \tag{4}$$

Defining

$$S_n(t) = \mathrm{sinc}(\omega_0(\gamma(t) - nT)) \tag{5}$$

equation (4) becomes

$$y(t) = \sum_{n=-\infty}^{\infty} h_n \int_{-\infty}^{\infty} S_n(s)\varphi(t-s)ds \tag{6}$$

The integral in (6) is just the convolution of $S_n(t)$ and $\varphi(t)$. Denoting it by $K_\gamma(n,t)$ yields

$$y(t) = \sum_{n=-\infty}^{\infty} h_n(S_n * \varphi)(t) = \sum_{n=-\infty}^{\infty} h_n K_\gamma(n,t) \tag{7}$$

where

$$K_\gamma(n,t) = \int_{-\infty}^{\infty} S_n(s)\varphi(t-s)ds \tag{8}$$

From equation (5), we notice that S_n depends on the bandwidth of f, ω_0, and values of the time warping function γ. Therefore, $K_\gamma(n,t)$ in equation (8) also depends on ω_0 and γ. This implies that $y(t)$ in equation (7) is only dependant on γ and ω_0, and not on other information about the underlying bandlimited signal f. This is stated more formally by the following property:

Property 1 *Given that the input to a linear time invariant filter is a time warped bandlimited signal $h = f \circ \gamma$ represented by a non-uniformly spaced sample sequence $\{h_n\}$, the output of the filter can be determined from the samples $\{h_n\}$ if one knows the time warping function γ and the bandwidth of f.*

III. AN EXAMPLE OF LOW-PASS FILTERING A NON-BANDLIMITED SIGNAL

In this section, an example of filtering a non-bandlimited signal is presented. The signal to be filtered has values

$$h(t) = \mathrm{sinc}(t^3/T)$$

That h is not bandlimited follows from the observation that it has greater than exponential growth when extended to the complex domain [5]. This signal may be obtained as the time warping of the bandlimited signal f with values

$$f(t) = \mathrm{sinc}(t/T)$$

by the monotone bijective function γ with values

$$\gamma(t) = t^3$$

The Nyquist sampling interval for f is $T = 0.125$ seconds, corresponding to a bandwidth of $\omega_0 = \pi/T \approx 25.133$ radians per second. A plot of the input signal $h(t)$ is shown in Figure 1.

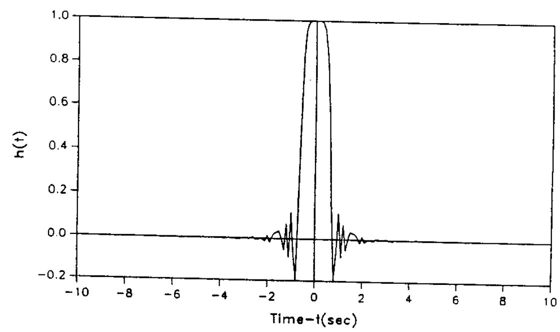

Figure 1: A time-domain plot of the non-bandlimited input signal values $h(t)$.

The filter to be implemented is an ideal low-pass with impulse response

$$\varphi(t) = \mathrm{sinc}((\omega_1/\pi)t)$$

where ω_1 is the bandwidth of the filter.

The first step in applying the method described in the previous section is to find $K_\gamma(n,t)$ of equation (8). Using equation (5) and the analytical expression of $\gamma(t)$ yields

$$S_n(t) = \mathrm{sinc}(\omega_0(t^3 - nT))$$

Then, equation (8) yields

$$K_\gamma(n,t) = \int_{-\infty}^{\infty} \frac{\sin(\omega_0(s^3 - nT))}{\omega_0(s^3 - nT)} \frac{\sin(\frac{\omega_1}{\pi}(t-s))}{\frac{\omega_1}{\pi}(t-s)} ds \tag{9}$$

Although the integrand in the above equation is too complicated to be integrated analytically, it can be evaluated approximately by a numerical integration technique due to Gauss, which is also called quadrature.

Numerical Integration

Let I be the integral defined by

$$I = \int_{x_1}^{x_2} f(x)dx \tag{10}$$

where x_1 and x_2 are finite and the explicit form of $f(x)$ is known but is sufficiently complicated that exact integration is intractible. An accurate quadrature formula for finding the value of the integral (10) was derived by Gauss and is based on Legendre polynomials [4]. This formula is called the Gaussian formula and is represented as follows:

$$\int_{x_1}^{x_2} f(x)dx = \sum_{i=1}^{N} \omega_i f(x_i) \tag{11}$$

where ω_i and x_i are called the weights and abscissas respectively. The weights are defined as:

$$\omega_i = \frac{2}{(1-x_i^2)(P'_N(x_i))^2}$$

where x_i are the abscissas of the N-point Gaussian quadrature formula (11) and they represent the roots of the related orthogonal Legendre polynomials $P_N(x)$.

In order to use the Gaussian formula (11) to approximate the integral for $K_\gamma(n,t)$ in (9), a change of variable that maps the infinite range of integration to finite limits is necessary. Note that this is due to the infinite impulse response (IIR) nature of the ideal lowpass filter; for a finite impulse response (FIR) filter, (9) would naturally reduce to an integral on a bounded interval. To accomplish a suitable variable change, denote the integrand by $g(s)$. Then the equation becomes

$$K_\gamma(n,t) = \int_{-\infty}^{0} g(s)ds + \int_{0}^{\infty} g(s)ds \tag{12}$$

Consider the first integral of (12)

$$\int_{-\infty}^{0} g(s)ds$$

Letting $x = e^s$ or $s = \ln x$, this transforms into

$$\int_0^1 \frac{g(\ln x)}{x} dx \tag{13}$$

In the same way, the change of variable $x = e^{-s}$ transforms the second integral of (12) into

$$\int_0^1 \frac{g(-\ln x)}{x} dx \tag{14}$$

Substituting (13) and (14) into (12) yields

$$K_\gamma(n,t) = \int_0^1 \frac{g(\ln x) + g(-\ln x)}{x} dx$$

Inserting the analytical expression for $g(s)$ and using the change of variables, the integral of (9) becomes

$$K_\gamma(n,t) =$$

$$\int_0^1 \frac{1}{x} \left[\frac{\sin(\omega_0((\ln x)^3 - nT))}{\omega_0((\ln x)^3 - nT)} \frac{\sin(\frac{\omega_1}{\pi}(t - \ln x))}{\frac{\omega_1}{\pi}(t - \ln x)} + \frac{\sin(\omega_0((-\ln x)^3 - nT))}{\omega_0((-\ln x)^3 - nT)} \frac{\sin(\frac{\omega_1}{\pi}(t + \ln x))}{\frac{\omega_1}{\pi}(t + \ln x)}) \right] dx \tag{15}$$

This integral may be approximated directly using the approach described above.

Computer Simulation

Once the integral $K_\gamma(n,t)$ of equation (15) is approximated, the output of the filter is determined by equation (7):

$$y(t) = \sum_{n=-\infty}^{\infty} h_n K_\gamma(n,t) \tag{16}$$

Since $\gamma(t) = t^3$, then $\gamma^{-1}(t) = t^{1/3}$ and

$$h_n = h(t_n) = h(\gamma^{-1}(nT)) = h((nT)^{1/3})$$

$$= \text{sinc}\left(\frac{((nT)^{1/3})^3}{T}\right) = \text{sinc}(n)$$

Suppose that there are altogether $2N_1 + 1$ samples of h. Then equation (16) becomes

$$y(t) = \sum_{n=-N_1}^{N_1} \text{sinc}(n) K_\gamma(n,t) \tag{17}$$

The output $y(t_1)$ at a particular time t_1 can be determined by finding $K_\gamma(n,t_1)$ of equation (15) for $n = -N_1, -N_1+1, \ldots\ldots, N_1 - 1, N_1$ and then using equation (17). The value of $y(t)$ at a given time t is determined by a computer program. The program asks for the bandwidth of f, the bandwidth of the filter, the number of sample points for Gauss-Legendre integration, and the limits of integration. This program, therefore, can be run for various filters and various signal bandwidths. The input parameters are shown in Table 1. The output of the filter, $y(t)$, is determined and shown in Figure 2 for the time values $-10 < t < 10$. Figures 3 and 4 show the spectrum of the input and output signals, respectively. It is apparent that the output of the filter is a bandlimited version of the input.

Table 1. Program input parameters.

Parameter	Symbol	Value
Bandwidth of f	ω_0	25.13 radians per second
Bandwidth of filter	ω_1	30 radians per second
Number of sample points	N_1	25
Number of sample points for Gauss-Legendre integration	N	120
Limits of integration	x_1	0
	x_2	1

IV. Conclusions

This paper shows that it is possible to apply a linear filter to certain non-bandlimited signals represented by sequences of non-uniformly spaced samples. Using the same notation as previously, if $h = f \circ \gamma$ is being the input to a linear filter, the output of the filter can be determined directly from the nonuniformly spaced samples h_n if one knows the analytical expression of γ and the bandwidth of f.

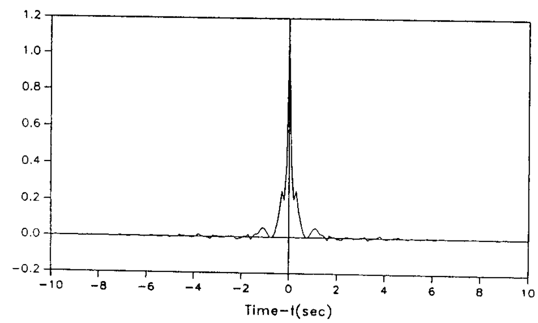

Figure 2: Time-domain output of the filter.

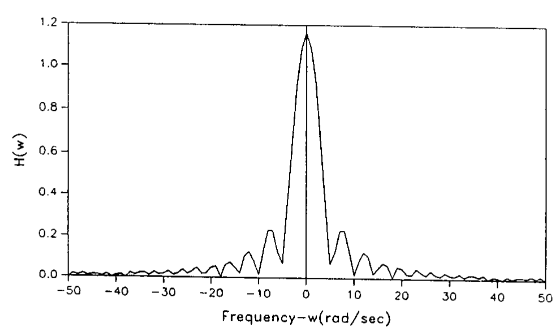

Figure 3: Spectrum of the input signal.

References

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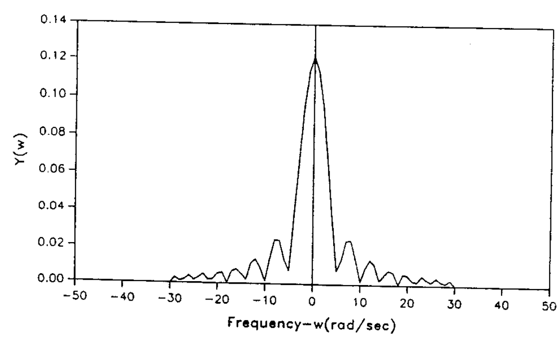

Figure 4: Spectrum of the output signal.