MULTIPLE-CHANNEL DETECTION USING GENERALIZED COHERENCE

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ABSTRACT

This paper examines the properties and performance of the Generalized Coherence (GC) Estimate as a statistic for detecting the presence of a common but unknown signal on several noisy channels. The H_0 distribution function of the GC estimate derived in earlier work is applied to develop detection thresholds corresponding to a range of false alarm probabilities for various numbers of channels and data sequence lengths. Results of simulations to determine the performance of a GC-based detector as a function of the signal-to-noise ratios on the data channels are also presented.

I. INTRODUCTION

In [7], the Generalized Coherence (GC) Estimate was introduced as a statistic for measuring the similarity of two or more complex-valued data sequences. This paper examines the application of the GC estimate in determining the presence of a common but unknown signal on several noisy data channels.

II. THE GENERALIZED COHERENCE ESTIMATE

Let $\mathbf{x}_m = (x_{m,1}, ..., x_{m,N})^T$ for $m = 1, ..., M$ be a collection of complex-valued data sequences, each of length N. The *generalized coherence estimate* formed from this collection is defined as

$$\gamma^2_{M,N}(\mathbf{x}_1, ..., \mathbf{x}_M) \triangleq 1 - \frac{g(\mathbf{x}_1, ..., \mathbf{x}_M)}{||\mathbf{x}_1||^2 \cdots ||\mathbf{x}_M||^2} \tag{1}$$

where $g(\mathbf{x}_1, ..., \mathbf{x}_M)$ denotes the determinant of the $M \times M$ *Gram matrix*

$$G(\mathbf{x}_1, ..., \mathbf{x}_M) \triangleq \begin{bmatrix} \langle \mathbf{x}_1, \mathbf{x}_1 \rangle & \cdots & \langle \mathbf{x}_1, \mathbf{x}_M \rangle \\ \vdots & & \vdots \\ \langle \mathbf{x}_M, \mathbf{x}_1 \rangle & \cdots & \langle \mathbf{x}_M, \mathbf{x}_M \rangle \end{bmatrix} \tag{2}$$

In these expressions, $\mathbf{x}^*$ denotes the complex conjugate of $\mathbf{x}$,

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle \triangleq \sum_{n=1}^{N} \mathbf{x}_{i,n} \mathbf{x}^*_{j,n} \tag{3}$$

is the inner product of $\mathbf{x}_i$ and $\mathbf{x}_j$, and

$$||\mathbf{x}_i||^2 \triangleq \langle \mathbf{x}_i, \mathbf{x}_i \rangle \tag{4}$$

is the squared magnitude of $\mathbf{x}$. Among the properties of the GC estimate are

1. $0 \leq \gamma^2(\mathbf{x}_1, ..., \mathbf{x}_M) \leq 1$
2. $\gamma^2(\mathbf{x}_1, ..., \mathbf{x}_M) = 1$ if and only if $\mathbf{x}_k = \sum_{j \neq k} \alpha_j \mathbf{x}_j$ for some set of complex scalars $\{\alpha_j\}$.
3. $\gamma^2(\mathbf{x}_1, ..., \mathbf{x}_M) = 0$ if and only if the $\mathbf{x}_k$ are all orthogonal.
4. For $M = 2$, the GC estimate $\gamma^2_{2,N}$ reduces to the well-known magnitude-squared coherence (MSC) estimate
$$\gamma^2(\mathbf{x}_1, \mathbf{x}_2) = \frac{|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle|^2}{||\mathbf{x}_1||^2 ||\mathbf{x}_2||^2} \tag{5}$$
5. $\gamma^2(\alpha_1 \mathbf{x}_1, ..., \alpha_M \mathbf{x}_M) = \gamma^2(\mathbf{x}_1, ..., \mathbf{x}_M)$ for any non-zero complex scalars $\alpha_1, ..., \alpha_M$ (ie., the GC estimate is invariant with respect to unknown channel gains).
6. $\gamma^2(\mathbf{x}_1, ..., \mathbf{x}_j, ..., \mathbf{x}_k, ..., \mathbf{x}_M) = \gamma^2(\mathbf{x}_1, ..., \mathbf{x}_k, ..., \mathbf{x}_j, ..., \mathbf{x}_M)$.

III. DISTRIBUTION OF THE GC ESTIMATE

A Gram-Schmidt procedure [8] was used in [7] to show that γ^2 can be factored into the form

$$\gamma^2(\mathbf{x}_1, ..., \mathbf{x}_M) = 1 - z_2 \cdots z_M \tag{6}$$

It was argued that, with the H_0 hypothesis that the $\mathbf{x}_m$ are independent Gaussian sequences, the factors z_m are independent beta-distributed random variables. In particular, with this H_0 hypothesis, z_m has the beta distribution $\beta[2N - 2(m-1), 2(m-1)]$ given in [1] by

$$Pr\{z_m \leq r\} = 1 - (1-r)^{N-1} \sum_{j=0}^{N-m} \binom{N-1}{j} \left[\frac{r}{1-r}\right]^j \tag{7}$$

for $2 \leq m \leq M$ and $0 \leq r \leq 1$.

Although this expression has a relatively simple polynomial structure, it is still a rather involved task to obtain a closed-form equation for the distribution of γ^2 from the distributions of the independent factors z_m. Equation (7) yields the recursion formula

$$\begin{cases} Pr\{z_2 \leq r\} = r^{N-1} \\ Pr\{z_{m+1} \leq r\} = Pr\{z_m \leq r\} + \binom{N-1}{N-m} r^{N-m}(1-r)^{m-1} \end{cases} \tag{8}$$

for fixed N and $2 \leq m \leq N$. This formula may be applied directly

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to determine the H_0 distribution function of the GC estimate for small numbers of channels. It also leads to another recursion formula that describes how the H_0 distribution function of $\gamma^2_{M+1,N}$ can be obtained from that of $\gamma^2_{M,N}$.

Distribution of the Two-Channel GC Estimate

The distribution function of the MSC estimate $\gamma^2 = \gamma^2_{2,N} = 1 - z_2$, as derived in [2] or [6], follows immediately from equation (8):

$$Pr\{\gamma^2 \le r\} = Pr\{1-\gamma^2 \ge 1-r\} = Pr\{z_2 \ge 1-r\} \quad (9)$$

$$= 1 - Pr\{z_2 \le 1-r\} = 1-(1-r)^{N-1} \quad (10)$$

Distribution of the Three-Channel GC Estimate

To calculate the distribution function of $\gamma^2_{3,N} = 1 - z_2 z_3$, note that the recursion formula (8) provides

$$Pr\{z_3 \le r\} = r^{N-1} + (N-1)r^{N-2}(1-r) \quad (11)$$

$$= (N-1)r^{N-2} - (N-2)r^{N-1} \quad (12)$$

Hence the joint density function $f_{z_2z_3}$ of the independent random variables z_2 and z_3 is

$$f_{z_2z_3}(a,b) = (N-1)^2(N-2)a^{N-2}[b^{N-3} - b^{N-2}] \quad (13)$$

and $Pr\{z_2z_3 \le r\}$ is given by the integral of $f_{z_2z_3}$ over the subset of the unit square in the $a-b$ plane on which $ab \le r$ (figure 1). Thus,

$$Pr\{z_2z_3 \le r\} = \quad (14)$$

$$\int_0^r \int_0^1 f_{z_2z_3}(a,b)\, db\, da + \int_r^1 \int_0^{r/a} f_{z_2z_3}(a,b)\, db\, da \quad (15)$$

$$= r^{N-1} + (N-1)(N-2)r^{N-1}\log(r) + (N-1)^2[r^{n-2} - r^{N-1}] \quad (16)$$

so that

$$Pr\{\gamma^2_{3,N} \le r\} = 1 - Pr\{z_2z_3 \le 1-r\} \quad (17)$$

$$= 1-(1-r)^{N-1} - (N-1)(N-2)(1-r)^{N-1}\log(1-r) \quad (18)$$

$$-(N-1)^2[(1-r)^{N-2} - (1-r)^{N-1}] \quad (19)$$

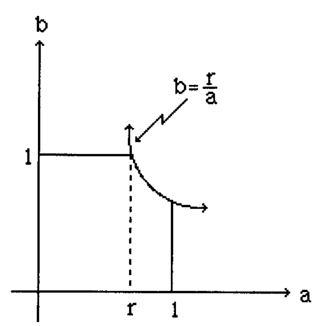

Figure 1: The distribution of the GC estimate for three channels is obtained by integrating the joint density function of z_2 and z_3 over the planar region pictured.

Recursion Formula for the Distribution of the Multi-Channel GC Estimate

Having derived the H_0 distribution functions of the GC estimate using $M = 2$ and $M = 3$ channels explicitly, a recursion formula for the distribution of the GC estimate using larger numbers of channels is now obtained as follows. Note that the independence of the random variables $\zeta_m \triangleq z_2z_3\cdots z_m$ and z_{m+1} implies that

$$Pr\{z_2z_3\cdots z_m z_{m+1} \le r\} = \quad (20)$$

$$\int_0^r \int_0^1 f_{\zeta_m}(a) f_{z_{m+1}}(b)\, db\, da + \int_r^1 \int_0^{r/a} f_{\zeta_m}(a) f_{z_{m+1}}(b)\, db\, da \quad (21)$$

$$= Pr\{\zeta_m \le r\} + \int_r^1 f_{\zeta_m}(a) Pr\{z_{m+1} \le r/a\}\, da \quad (22)$$

Using the recursion formula for the distribution of z_{m+1} from equation (8) above, this expression yields

$$Pr\{\zeta_{m+1} \le r\} = Pr\{\zeta_m \le r\} + \int_r^1 f_{\zeta_m}(a)\, Pr\{z_m \le r/a\}\, da \quad (23)$$

$$+ \binom{N-1}{N-m} \int_r^1 f_{\zeta_m}(a) \left(\frac{r}{a}\right)^{N-m} \left(1-\frac{r}{a}\right)^{m-1} da \quad (24)$$

Since $Pr\{\gamma^2_{m+1,N} \le r\} = 1 - Pr\{\zeta_{m+1} \le 1-r\}$, this expression describes how the H_0 distribution function of $\gamma^2_{m+1,N}$ can be obtained from that of $\gamma^2_{m,N}$.

IV. PROBABILITY OF FALSE ALARM THRESHOLDS

In the context of multiple-channel signal detection, the principal value of knowing the H_0 distribution function of the GC estimate is its utility in determining detection thresholds corresponding to desired false alarm probabilities. Such thresholds provide the basis for deciding whether to hypothesize the presence or absence of a common signal on the noisy channels given a GC value computed using actual data sequences from the channels. Tables 1 and 2 give values of the GC estimate corresponding to a range of probabilities of false alarm P_f and sample sequence lengths N for two and three channels, respectively.

V. DETECTION PERFORMANCE

Using the detection thresholds for $\gamma^2_{2,N}$ given in table 1, simulations were run to evaluate the ability of a GC-based detector to identify the presence a common signal having given signal-to-noise ratios (SNR's) on two noisy channels. Similar simulations were performed for the three-channel case using the detection thresholds given in table 2. The procedure used in these simulations was as follows:

1. A desired level of probability of false alarm P_f was chosen and the GC value corresponding to that P_f was determined (from the above tables or directly from the distribution function of the GC estimate).
2. A desired level of probability of detection P_d was chosen.
3. Channels containing a pseudorandom signal sequence S and independent pseudorandom noise sequences $N_1, \ldots, N_M$ were blended to obtain M channels of a common signal in noise having known SNR's.

P_f	N=4	N=8	N=16	N=32	N=64	N=128	N=256
10^{-1}	.5358411	.2803143	.1423041	.0715855	.0358891	.0179672	.0089891
10^{-2}	.7845565	.4820525	.2643577	.1380464	.0704902	.0356116	.0178974
10^{-3}	.9000000	.6272406	.3690427	.1997498	.1038495	.0529390	.0267256
10^{-4}	.9535841	.7317304	.4588305	.2570360	.1360116	.0699551	.0354745
10^{-5}	.9784557	.8069302	.5358411	.3102215	.1670193	.0866654	.0441447
10^{-6}	.9900000	.8610505	.6018928	.3595996	.1969143	.1030755	.0527370

Table 1: Values of $\gamma_{2,N}^2$ corresponding to certain probabilities of false alarm.

P_f	N=4	N=8	N=16	N=32	N=64	N=128	N=256
10^{-1}	.8754568	.5521470	.3046456	.1593690	.0814290	.0411485	.0206850
10^{-2}	.9646636	.7195777	.4367786	.2398269	.1255404	.0642102	.0324684
10^{-3}	.9892299	.8176667	.5356765	.3067238	.1640644	.0848355	.0431347
10^{-4}	.9966401	.8793963	.6140143	.3651875	.1993141	.1041324	.0532235
10^{-5}	.9989429	.9194506	.6775521	.4173248	.2321731	.1225105	.0629335
10^{-6}	.9996663	.9458743	.7297485	.4643189	.2631002	.1401745	.0723633

Table 2: Values of $\gamma_{3,N}^2$ corresponding to certain probabilities of false alarm.

P_f	N=4	8	16	32	64	128	256
10^{-1}	1.2	-0.3	-1.5	-2.6	-3.6	-4.3	-5.2
10^{-2}	3.6	1.4	-0.1	-1.4	-2.5	-3.2	-4.1
10^{-3}	5.5	2.5	0.8	-0.7	-1.8	-2.7	-3.6
10^{-4}	7.3	3.5	1.4	-0.1	-1.4	-2.2	-3.2
10^{-5}	9.0	4.4	2.0	0.3	-0.9	-1.9	-2.9
10^{-6}	10.7	5.2	2.5	0.7	-0.6	-1.6	-2.6

Table 3: Equal-channel SNR values (dB) necessary to achieve a 50% detection probability using a two-channel GC detector (from simulations).

P_f	N=4	8	16	32	64	128	256
10^{-1}	1.0	-0.8	-2.2	-3.5	-4.1	-4.9	-6.1
10^{-2}	3.0	0.5	-1.0	-2.1	-3.2	-4.1	-5.0
10^{-3}	4.6	1.4	-0.3	-1.5	-2.7	-3.6	-4.3
10^{-4}	6.0	2.1	0.3	-1.1	-2.3	-3.2	-4.2
10^{-5}	7.3	2.8	0.7	-0.8	-2.0	-2.9	-3.9
10^{-6}	8.6	3.4	1.1	-0.5	-1.7	-2.8	-3.5

Table 4: Equal-channel SNR values (dB) necessary to achieve a 50% detection probability using a three-channel GC detector (from simulations).

4. The GC estimate was computed using a large number of independent subsequences from the M synthetic data sequences. The ratio of the number of computed GC values exceeding the detection threshold to the total number of values (ie., the *frequency of detection* F_d) was compared to the desired P_d level. When they were equal, a sequence of SNR's $(SNR_1, ..., SNR_M)$ corresponding the the desired (P_d, P_f) pair was recorded; otherwise the SNR's on the channels were adjusted and this step was repeated.

Table 3 gives SNR values which, if equal on both channels, yield a detection frequency of 50% for a two-channel GC detector. These values are given for a range of false alarm probability levels P_f and sequence lengths N. Table 4 gives equal-channel SNR's that yield a 50% detection frequency for a three-channel GC detector.

Results involving unequal SNR's on the data channels are summarized in figure 2 for $N = 64$. The family of solid curves represents results from three-channel GC detectors. Each point on one of these curves corresponds to a triplet (SNR_1, SNR_2, SNR_3) of SNR values that yield a fixed detection performance ($P_d = 0.5$, $P_f = 10^{-3}$). The three SNR values (in dB) are obtained from the horizontal axis (SNR_1), the vertical axis (SNR_2), and the curve label (SNR_3). Note that a certain symmetry is to be expected due to the interchangability of channels (Property 6 in Section II, above).

The single dashed curve in figure 2 shows the SNR's needed to obtain this same level of detection performance ($P_d = 0.5$, $P_f = 10^{-3}$) using a two-channel GC detector. For this curve, the two SNR's are simply read off the axes.

Discussion of Performance Results

Because the GC estimate reduces to the MSC estimate for $M = 2$, the two-channel GC detector is actually a MSC detector. The performance of the MSC detector has been studied rather extensively (see [4] or [9], for example), and the results presented here correspond closely to those obtained in other work.

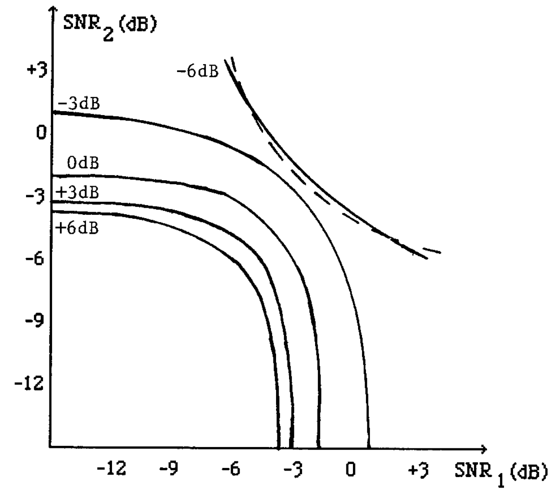

Figure 2: Detection curves for two-channel (dashed) and three-channel (solid) GC detectors. These curves show the SNR's necessary to achieve detection performance ($P_d = 0.5$, $P_f = 10^{-3}$) with $N = 64$.

The curves in figure 2 invite a few observations:

1. The use of a third channel, when compared to doing two-channel MSC detection, is of significant value when the SNR on the third channel is high. In the case shown, detection performance improves if SNR_3 is greater than about -6 dB; is about the same as two-channel performance if $SNR_3 = -6$ dB; and may actually be worse than two-channel performance when SNR_3 is very low.
2. The shapes of the three-channel curves for $SNR_3 \geq -3$ dB indicate that detection is possible if no signal is present on one of the three channels provided that the SNR's on the other two channels are sufficiently high (note the asymptotic behavior of these curves as SNR_1 or SNR_2 gets very small). This phenomenon, which can be predicted from the geometrical structure of the GC estimate [3], may be a feature in some applications and a drawback in others.
3. The different shape of the solid curves for $SNR_3 \leq -6$ dB is a result of the phenomenon just discussed; on these curves detection occurs primarily due to the high SNR's on channels 1 and 2.

These results also indicate that cases exist in which a legitimate three-channel detection can be obtained, but a pair of two-channel detections cannot. This can be of significant value in source localization.

VI. CONCLUDING REMARKS

The authors believe the GC estimate has numerous appealing properties as a nonparametric multiple-channel detection statistic, several of which are lacking in other statistics (such as *Multiple Coherence* [9]) often used in multiple-channel detection applications. Some of these are pointed out in [3].

Preliminary performance results presented in this paper suggest that detection schemes based on the GC estimate may prove valuable in certain situations, especially those in which moderately strong signals on some channels provide a means to detect the presence of weak signals on other channels. It should be noted, however, that we have not yet carefully compared performance results obtained using generalized coherence with those published for other techniques. Moreover, the results presented in this paper were obtained using simulated data for which the assumptions underlying the derivation of detection thresholds (independence, Gaussian distributions, etc.) are valid by design. Performance tests using real data from an actual application – for which the assumptions will be only approximately correct at best – have not yet been attempted for more than two data channels.

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