# GENERALIZED COHERENCE

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**Abstract**

This paper introduces a test for detecting the presence of a common signal on $M$ noisy channels that generalizes a popular technique using the magnitude-squared coherence (MSC) estimate for detecting a common signal on two channels. The basis of this test is the *generalized coherence estimate*, a statistic involving the normalized $M \times M$ Gram matrix determinant, which reduces to the MSC estimate for $M$=2. The distribution of the generalized coherence estimate is derived under the assumptions of independent noise and no common signal on the $M$ channels.

## 1. INTRODUCTION

The ability to determine whether a common signal is present on two or more noisy channels is desirable for a variety of applications, particularly in situations where a signal source is to be detected and localized from measurements made at several sensors. In the case of two channels, a popular test is to compare the magnitude-squared coherence (MSC) estimate computed using sample sequences from the two channels to a threshold [2]. This is an especially useful test because the statistical behavior of the MSC estimate is well understood in the absence of common signal, allowing thresholds corresponding to particular false alarm probabilities to be readily established. Recent work has shown that the MSC estimate has a geometrical interpretation in light of which its statistical properties can be derived in a direct and intuitive way [4].

In this paper, a test is introduced for detecting a common signal on $M$ channels that generalizes the MSC approach in a natural way. This test is based on the *generalized coherence estimate*, a statistic involving the normalized determinant of an $M \times M$ Gram matrix. Section 2 discusses the MSC estimate and formulates it in terms of a two-by-two Gram matrix. In section 3, the distribution of the MSC estimate is derived using properties of the Gram matrix determinant. Section 4 extends the ideas of section 2 to define the $M$-dimensional generalized coherence estimate, the distribution of which is derived in section 5. Section 6 contains a discussion of the geometrical character inherent in this type of detection statistic and describes some ideas for future work in this vein.

## 2. THE MSC ESTIMATE

Given two complex sequences $\mathbf{x}_1 \triangleq \{x_{1,1}, \ldots, x_{1,N}\}$ and $\mathbf{x}_2 \triangleq \{x_{2,1}, \ldots, x_{2,N}\}$, the MSC estimate is given by

$$\gamma^2(\mathbf{x}_1,\mathbf{x}_2) \triangleq \frac{|<\mathbf{x}_1,\mathbf{x}_2>|^2}{||\mathbf{x}_1||^2||\mathbf{x}_2||^2} \quad (1)$$

where $<\mathbf{x}_1,\mathbf{x}_2>$ is the inner product of $\mathbf{x}_1$ and $\mathbf{x}_2$, defined by

$$<\mathbf{x}_1,\mathbf{x}_2> \triangleq \sum_{i=1}^{N} x_{1,i}\, x_{2,i}^* \quad (2)$$

In these expressions, * denotes complex conjugation and $||\mathbf{x}||^2 \triangleq <\mathbf{x},\mathbf{x}>$ is the squared magnitude of $\mathbf{x}$.

By the Schwarz inequality [7], the MSC estimate has the property that

$$0 \le \gamma^2(\mathbf{x}_1,\mathbf{x}_2) \le 1$$

for any sequences $\mathbf{x}_1$ and $\mathbf{x}_2$. In particular, $\gamma^2(\mathbf{x}_1,\mathbf{x}_2) = 0$ if $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal and $\gamma^2(\mathbf{x}_1,\mathbf{x}_2) = 1$ when $\mathbf{x}_1 = a\mathbf{x}_2$ for any non-zero complex number $a$.

The MSC estimate is particularly useful as a detection statistic because its distribution function is known explicitly in the case that no common signal component is present and the two sequences are statistically independent Gaussian noise. This allows thresholds corresponding to particular probabilities of false alarm to be computed directly. The distribution of the MSC estimate is given by

$$Pr\,\{\gamma^2 \le r\} = 1-(1-r)^{N-1} \quad 0 \le r \le 1 \quad (3)$$

(see [4] or [2]).

For the purpose of generalization to $M$ dimensions, it is useful to consider an equivalent formulation of the MSC estimate. The MSC estimate may be written as

$$\gamma^2(\mathbf{x}_1,\mathbf{x}_2) = 1 - \frac{g(\mathbf{x}_1,\mathbf{x}_2)}{||\mathbf{x}_1||^2||\mathbf{x}_2||^2} \quad (4)$$

where

$$g(\mathbf{x}_1,\mathbf{x}_2) = det \begin{bmatrix} <\mathbf{x}_1,\mathbf{x}_1> & <\mathbf{x}_1,\mathbf{x}_2> \\ <\mathbf{x}_2,\mathbf{x}_1> & <\mathbf{x}_2,\mathbf{x}_2> \end{bmatrix}. \quad (5)$$

This matrix of inner products is called a *Gram matrix* [3] and its determinant known as a *Gram determinant*. The Gram determinant $g$ has the property that

$$0 \le g(\mathbf{x}_1,\mathbf{x}_2) \le ||\mathbf{x}_1||^2\, ||\mathbf{x}_2||^2 \quad (6)$$

for any sequences $\mathbf{x}_1$ and $\mathbf{x}_2$. In particular, $g(\mathbf{x}_1,\mathbf{x}_2) = 0$ if $\mathbf{x}_1$ and $\mathbf{x}_2$



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are proportional and $g(\mathbf{x}_1,\mathbf{x}_2) = \|\mathbf{x}_1\|^2\,\|\mathbf{x}_2\|^2$ when $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Other properties of the Gram determinant include

$$g(\mathbf{x}_1,\mathbf{x}_2) = g(\mathbf{x}_2,\mathbf{x}_1) \tag{7}$$

and

$$g(\mathbf{x}_1,\mathbf{x}_2 + a\mathbf{x}_1) = g(\mathbf{x}_1,\mathbf{x}_2) \tag{8}$$

for any complex number $a$. These properties imply that the Gram determinant is only sensitive to those components of its arguments that are orthogonal. With this in mind, assume $\mathbf{x}_1$ and $\mathbf{x}_2$ are non-zero and consider two new sequences

$$\mathbf{u}_1 \triangleq \mathbf{x}_1 \tag{9}$$

and

$$\mathbf{u}_2 \triangleq \mathbf{x}_2 - \left[\frac{<\mathbf{x}_1,\mathbf{x}_2>}{\|\mathbf{x}_1\|^2}\right]\mathbf{x}_1$$

These sequences are orthogonal, so that

$$\begin{aligned} g(\mathbf{x}_1,\mathbf{x}_2) &= g(\mathbf{u}_1,\mathbf{u}_2) \\ &= \|\mathbf{u}_1\|^2\,\|\mathbf{u}_2\|^2 \\ &= \|\mathbf{x}_1\|^2\,\|\mathbf{x}_2 - \left[\frac{<\mathbf{x}_1,\mathbf{x}_2>}{<\mathbf{x}_1,\mathbf{x}_1>}\right]\mathbf{x}_1\|^2 \end{aligned} \tag{10}$$

This method of obtaining a set of orthogonal vectors $\{\mathbf{u}_1,\mathbf{u}_2\}$ with the same Gram determinant as the original set $\{\mathbf{x}_1,\mathbf{x}_2\}$ is known as the *Gram-Schmidt procedure* [7]. The vector $\mathbf{u}_2$ is the component of $\mathbf{x}_2$ that is orthogonal to $\mathbf{x}_1$. Hence, the Gram determinant is the product of two terms: the first, $\|\mathbf{u}_1\|^2 = \|\mathbf{x}_1\|^2$ is the squared length of $\mathbf{x}_1$ and the second, $\|\mathbf{u}_2\|^2$, is the squared length of the component of $\mathbf{x}_2$ that is orthogonal to $\mathbf{x}_1$.

## 3. THE DISTRIBUTION OF THE MSC ESTIMATE

As pointed out above, the ability to calculate the distribution of the MSC estimate when no common signal is present and the sequences are from statistically independent Gaussian distributions is important to the utility of the MSC estimator. In this section the distribution function of $\gamma^2$ is derived in a way that will extend naturally to the case of the generalized coherence of $M$ sequences in the next two sections.

As above, the MSC estimate may be written as

$$\gamma^2(\mathbf{x}_1,\mathbf{x}_2) = 1 - z \tag{11}$$

where

$$z \triangleq \frac{g(\mathbf{x}_1,\mathbf{x}_2)}{\|\mathbf{x}_1\|^2\|\mathbf{x}_2\|^2} = \frac{<\mathbf{u}_2,\mathbf{u}_2>}{\|\mathbf{x}_2\|^2} \tag{12}$$

Denote by $P(\mathbf{x}_2|\mathbf{x}_1)$ the projection of $\mathbf{x}_2$ onto the subspace spanned by $\mathbf{x}_1$. Then

$$\mathbf{x}_2 = P(\mathbf{x}_2|\mathbf{x}_1) + \mathbf{u}_2 \tag{13}$$

and, since $P(\mathbf{x}_2|\mathbf{x}_1)$ and $\mathbf{u}_2$ are orthogonal, the Pythagorean theorem implies

$$\|\mathbf{x}_2\|^2 = \|P(\mathbf{x}_2|\mathbf{x}_1)\|^2 + \|\mathbf{u}_2\|^2. \tag{14}$$

In order to calculate the distribution of $z$ under the assumption that $\mathbf{x}_1$ and $\mathbf{x}_2$ are complex Gaussian white noise sequences, $z$ must be expressed in terms of of the components of the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Results presented in [4] show that the squared length of the projection of a complex vector of length $N$ onto another complex $N$-vector is equal to the squared length of the projection of a real $2N$-vector onto a real two-dimensional subspace. The real and imaginary components of the complex vector $\mathbf{x}_2$ define the $2N$-dimensional real vector, $\mathbf{x}_{2R}$ that is projected onto a two-dimensional subspace of $\mathrm{R}^{2N}$ that is determined by the complex vector $\mathbf{x}_1$. If $\mathbf{x}_2$ has statistically independent components, each with mean zero and variance $\sigma^2$, then so will $\mathbf{x}_{2R}$. Using this construction it is possible, with appropriate change of basis, to write

$$\|\mathbf{x}_2\|^2 = \sum_{i=1}^{2N} v_i^2 = \|\mathbf{x}_{2R}\|^2 \tag{15}$$

and

$$\|P(\mathbf{x}_2|\mathbf{x}_1)\|^2 = \sum_{i=1}^{2} v_i^2 \tag{16}$$

The expression (16) contains only the two terms arising from the projection of $\mathbf{x}_{2R}$ onto a two-dimensional subspace. Hence, from (14),

$$\|\mathbf{u}_2\|^2 = \sum_{i=3}^{2N} v_i^2 \tag{17}$$

where the $v_i$ are statistically independent, zero-mean, Gaussian random variables with common variance $\sigma^2$. Thus,

$$z = \frac{\sum_{i=3}^{2N} v_i^2}{\sum_{i=1}^{2N} v_i^2} \tag{18}$$

which is known [1] to have a beta distribution, $\beta(2N-2,2)$, of $2N-2$ and 2 degrees of freedom. Furthermore,

$$\gamma^2(\mathbf{x}_1,\mathbf{x}_2) = 1 - z = \frac{\sum_{i=1}^{2} v_i^2}{\sum_{i=1}^{2N} v_i^2} \tag{19}$$

which has a $\beta(2,2N-2)$ distribution. This distribution is expressed explicitly by equation (3) above.

## 4. THE GENERALIZED COHERENCE ESTIMATE

Consider now the situation in which the similarity, or coherence, of $M$ complex sequences $\mathbf{x}_1,\ldots,\mathbf{x}_M$ is to be measured. The above analysis suggests that the MSC estimate can be naturally extended to the case of $M$ non-zero sequences if a *generalized coherence estimate* $\gamma^2(\mathbf{x}_1,\ldots,\mathbf{x}_M)$ is defined by

$$\gamma^2(\mathbf{x}_1,\ldots,\mathbf{x}_M) = 1 - \frac{g(\mathbf{x}_1,\ldots,\mathbf{x}_M)}{\|\mathbf{x}_1\|^2 \cdots \|\mathbf{x}_m\|^2} \tag{20}$$

where $g(\mathbf{x}_1,\ldots,\mathbf{x}_M)$ is the determinant of an $M \times M$ Gram matrix whose $(i,j)^{\text{th}}$ element is $<\mathbf{x}_i,\mathbf{x}_j>$.

The Gram determinant $g$ for $M$ sequences has the properties

$$0 \le g(\mathbf{x}_1,...,\mathbf{x}_M) \le \prod_{i=1}^{M} \|\mathbf{x}_i\|^2, \qquad (21)$$

$$g(\mathbf{x}_1,...,\mathbf{x}_k + a\mathbf{x}_j,...,\mathbf{x}_M) = g(\mathbf{x}_1,...,\mathbf{x}_k,...,\mathbf{x}_M) \quad j \ne k \qquad (22)$$

and

$$g(\mathbf{x}_1,...,\mathbf{x}_j,\mathbf{x}_k,...,\mathbf{x}_M) = g(\mathbf{x}_1,...,\mathbf{x}_k,\mathbf{x}_j,...,\mathbf{x}_M) \qquad (23)$$

for any sequences $\mathbf{x}_1,...,\mathbf{x}_M$. As before, $g$ achieves its maximum value $\prod_{i=1}^{M} \|\mathbf{x}_i\|^2$ when the sequences are orthogonal and its minimum value of zero when the sequences are linearly dependent, i.e., when they span a subspace of dimension less than $M$. This implies that the generalized coherence will again have values between zero and one. Also, the property described by equation (22) shows that the contribution of a sequence to the value of $g$ depends only on its components that are linearly independent of all the other sequences.

## 5. THE DISTRIBUTION OF GENERALIZED COHERENCE

In this section the approach used in Section 3 to derive the distribution of the MSC estimate is extended to determine the distribution of the generalized coherence estimate. Consider an orthogonal basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_M\}$ for the $M$ sequences obtained via the Gram-Schmidt procedure:

$$\mathbf{u}_1 \triangleq \mathbf{x}_1 \qquad (24)$$

$$\mathbf{u}_j \triangleq \mathbf{x}_j - P(\mathbf{x}_j|\mathbf{x}_1,..,\mathbf{x}_{j-1}) \qquad j=2,..M \qquad (25)$$

where $P$ projects $\mathbf{x}_j$ onto the linear subspace spanned by $\{\mathbf{x}_1,...,\mathbf{x}_{j-1}\}$. Define

$$z \triangleq \frac{g(\mathbf{x}_1,...,\mathbf{x}_M)}{\|\mathbf{x}_1\|^2...\|\mathbf{x}_M\|^2} \qquad (26)$$

Then from the properties of the Gram determinant

$$z = \prod_{j=2}^{M} \frac{\|\mathbf{u}_j\|^2}{\|\mathbf{x}_j\|^2} \qquad (27)$$

Hence $z$ is the product of $M-1$ terms $z_2,...,z_M$ with the $j$ th term being of the form

$$z_j \triangleq \frac{\|\mathbf{u}_j\|^2}{\|\mathbf{x}_j\|^2} \qquad j=2,..,M \qquad (28)$$

where $\|\mathbf{u}_j\|^2$ is the squared distance from $\mathbf{x}_j$ to the space spanned by $\{\mathbf{x}_1,\mathbf{x}_2,..,\mathbf{x}_{j-1}\}$. In terms of real vectors the space spanned by $\{\mathbf{x}_1,..,\mathbf{x}_{j-1}\}$ will have dimension $2(j-1)$ and its orthogonal complement will have dimension $2N-2(j-1)$. Hence the argument given in section 3 for the MSC estimate extends directly to this $M$-dimensional setting. In particular, $z_j$ has a beta distribution of $2N-2(j-1)$ and $2(j-1)$ degrees of freedom. Thus the generalized coherence estimate is equal to 1 minus the product of $M-1$ independent variates, each of which has a beta distribution. Using [1] these distributions are given by

$$Pr\{z_j \le r\} = 1-(1-r)^{N-1}\sum_{i=0}^{N-j}\binom{N-1}{i}\left[\frac{r}{1-r}\right]^i \quad 2\le j\le M \quad 0\le r\le 1 \qquad (29)$$

The relatively simple polynomial character of these terms should make the calculation of the the distribution of $z$ a relatively straightforward, if tedious, convolution of the distributions of the $z_j$.

## 6. DISCUSSION

The approach we have suggested for generalizing the idea of coherence to multiple sequences is based on the Gram determinant. This determinant, in effect, measures the dissimilarity of the signal set. The subsequent normalization and subtraction from unity converts this dissimilarity measure to a similarity or coherence measure with values between zero and one.

If the vectors $\mathbf{x}_1,..,\mathbf{x}_M$ were real, then the associated Gram determinant would be what is called the *generalized variance* in the statistics literature (see [9]). When the vectors are real, the Gram determinant has a geometric interpretation as the square of the volume of a parallelotope in $M$ dimensions (i.e. the $M$-dimensional version of a parallelogram), with edges determined by the vectors. When the vectors are orthogonal, the paralleltope has orthogonal edges and therefore maximum volume. For complex vectors, the geometric interpretation becomes more difficult but the basic notions are still present.

The method presented in this paper for deriving the distribution of the generalized coherence is based on the geometric properties of the Gram determinant, interpretation of the complex projection in terms of real vectors, and knowledge of the beta distribution. We have also explored (but omitted from this note) an alternative approach which is more analytical in character and yields the same results. This alternative approach is based on the works of Goodman ([5], [6]).

The next steps with regard to this work that would be of interest are finding the gains in signal detection performance that can be achieved by the use of multiple signal sequences using the generalized coherence function and to compare it with other approaches, such as the *multiple coherence* [8], for dealing with multiple signals. In addition, it should be possible to derive the distribution of the generalized coherence estimate under the assumption that the sequences have spherically symmetric distributions -- a slightly weaker condition than the Gaussian distributions assumed in this note. Such a derivation for the MSC estimate was given in [4].

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