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A CONSEQUENCE OF SIGNAL NORMALIZATION IN SPECTRUM ANALYSIS

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Abstract

Normalization of signals is used as a means of dynamic range regulation in many applications, but the inherent non-linear nature of normalization and its interaction with other commonly used tools of signal processing are often ignored. This paper presents some of the issues involved in signal normalization and examines the role of normalization in the context of spectrum analysis. In particular, the question of the relative significance of spectral phase and magnitude is considered for normalized signals.

1. INTRODUCTION

In a wide variety of applications, the dynamic range of signals is regulated by normalization or by gain control schemes involving normalization in conjunction with windowing. Effects of signal normalization are seldom considered because many important properties of signals are scale invariant. Thus, normalization is often regarded as simple multiplication by a constant. While valid in many situations, this point of view has led to some neglect of the role normalization can play in signal processing systems. This paper points out one effect of normalization in the context of a well-established topic in spectrum analysis: the relative significance of (Fourier) magnitude and phase in signals.

2. NORMALIZATION OF SIGNALS

In order to speak of normalization of signals it is necessary to have the concept of a norm. It will serve to regard signals as elements in a real or complex Banach space S with norm η [6]. For all non-zero $\phi \in S$, the signal $N(\phi) \triangleq \phi/\eta(\phi)$ is a normalized version of ϕ.

In practical terms, for a signal $\phi:\mathbf{R}^n \rightarrow \mathbf{C}$, η might be the energy in the signal

$$\eta(\phi) = \int_{\mathbf{R}^n} |\phi(t)|^2 \, dt$$

or its maximum magnitude

$$\eta(\phi) = \sup_{t \in \mathbf{R}^n} |\phi(t)|,$$

or some other suitable measure of the "size" of the signal that is meaningful within the context of the application.

If S_1 and S_2 are Banach spaces of signals, then a signal processing operation taking signals in S_1 as inputs and producing signals in S_2 as outputs may be regarded as a function $T:S_1 \rightarrow S_2$. A signal processing operation is linear if it is linear as a mapping of Banach spaces: that is, if $T(\alpha\phi+\beta\psi) = \alpha T(\phi)+\beta T(\psi)$ for all $\phi,\psi \in S_1$ and all scalars α and β. In this sense, normalization N is a non-linear signal processing operation, as is demonstrated by noting

$$N(2\phi) = \frac{2\phi}{\eta(2\phi)} = \frac{2\phi}{2\eta(\phi)} = \frac{\phi}{\eta(\phi)} \neq 2N(\phi)$$

Moreover, it can be argued that normalization is a "highly non-linear" operation because of inherent difficulties in approximating it even locally by a linear operation [2].

3. THE MAGNITUDE VERSUS PHASE PROBLEM

Over the past two decades, an impressive engineering folklore as well as a considerable body of literature has grown up around the question of the relative significance of phase and magnitude in signals. Much of the research in this area has addressed two particular problems: establishing conditions under which a signal is specified to certain degrees by either its magnitude or phase spectrum, and reconstructing signals from magnitude or phase information. Historical perspective and the current state of research regarding these questions are presented in [1], [4], [5], and [8].

In this section, the magnitude-versus-phase question is considered for normalized signals. A statistical perspective in which signals are regarded as realizations of stochastic processes is taken, thereby allowing a probabilistic examination of the distortion introduced by loss or corruption of either phase or magnitude information.

Consider a periodic, finite-energy (L^2) signal represented by a Fourier series

$$\phi(t) = \sum c_m e^{i(mt+\theta_m)}$$

in which the magnitude coefficients $c_m \geq 0$ and phase coefficients $\theta_m \in [0,2\pi]$ appear explicitly. In order to regulate its energy within one period, such a signal might be normalized by its L^2 norm [6]

$$\eta(\phi) = \|\phi\| \triangleq \left[\int |\phi(t)|^2 \, dt \right]^{1/2} = \left[\sum c_m^2 \right]^{1/2}$$

yielding a new signal $\hat{\phi}$ with values $\hat{\phi}(t) = \phi(t)/\|\phi\|$.

3.1 Sensitivity to Perturbation of Coefficients

To address the question of sensitivity of the signal values to small perturbations in the magnitude and phase coefficients, the mean-square values of the partial derivatives of the signal with respect to each type of coefficient may be compared. The phase coefficients θ_m are assumed independent and uniform on $[0,2\pi]$. In order that the values of the derivatives be directly comparable, it is necessary to use log-polar representation in the Fourier series:

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$$\phi(t) = \sum e^{b_m} e^{i(mt+\theta_m)}$$

Direct computation yields

$$\frac{\partial\hat{\phi}}{\partial\theta_k} = \frac{i e^{b_k} e^{i(kt+\theta_k)}}{(\sum e^{2b_m})^{1/2}}$$

so that

$$\left|\frac{\partial\hat{\phi}}{\partial\theta_k}\right|^2 = \frac{e^{2b_k}}{\sum e^{2b_m}} = \frac{c_k^2}{\sum c_m^2}$$

The derivative of ϕ with respect to a log-magnitude coefficient is given by

$$\frac{\partial\hat{\phi}}{\partial b_k} = \frac{\left[(e^{b_k} e^{i(kt+\theta_k)} \sum e^{2b_m}) - (e^{2b_k} \sum e^{b_m} e^{i(mt+\theta_m)})\right]}{(\sum e^{2b_m})^{-3/2}}$$

$$= (\sum c_m^2)^{-3/2}\left[(c_k e^{i(kt+\theta_k)} \sum c_m^2) - (c_k^2 \sum c_m e^{i(mt+\theta_m)})\right]$$

Under the assumptions on the θ_k, this leads to

$$E\left|\frac{\partial\hat{\phi}}{\partial b_k}\right|^2 = \frac{c_k^2 \sum c_m^2 - c_k^4}{(\sum c_m^2)^2}$$

Combining the above yields the conclusion

$$E\left|\frac{\partial\hat{\phi}}{\partial b_k}\right|^2 \le \left|\frac{\partial\hat{\phi}}{\partial\theta_k}\right|^2$$

Thus, a small change to a magnitude coefficient of such a normalized signal has less expected effect on the signal values than an equivalently small change to a phase coefficient. In this sense, phase is a more critical component of normalized signals.

3.2 Distortion Due to Loss of Phase or Magnitude

In [5], some striking results are shown that suggest loss of phase information causes far more distortion to two-dimensional images than does loss of magnitude information. One of the phenomena demonstrated was the following. The two-dimensional discrete Fourier transform (DFT) of an image $\phi(x,y)$ was computed yielding $\Phi(\omega,\nu)$. Two new signals

$$\Phi_p(\omega,\nu) \triangleq \frac{\Phi(\omega,\nu)}{|\Phi(\omega,\nu)|}$$

and

$$\Phi_m(\omega,\nu) \triangleq |\Phi(\omega,\nu)|$$

were computed from Φ Note that the phase of Φ_p at each point (ω,ν) is identical to that of Φ, but the magnitude is one. Similarly, the respective magnitudes of Φ_m and Φ are the same at each point, but the phase of Φ_m is identically zero. The inverse DFTs of Φ_m and Φ_p were computed to obtain images with values $\phi_m(x,y)$ and $\phi_p(x,y)$, respectively. These images were, presumably, scaled to an appropriate level for display on some device, displayed, and compared by eye to the original.

Figures 1b and 1c were obtained (as ϕ_m and ϕ_p, respectively) from figure 1a by the procedure just described. Each of these images was normalized by a fixed constant multiple of its L^2 norm in order to scale it to an appropriate level for display. They demonstrate the same phenomenon observed in [5]: subjectively, at least, the phase-only image resembles the original image far more than does the magnitude-only image.

3.2.1 Phase-Loss Distortion in a Two-Coefficient Signal

To motivate a more general analysis of the distortion introduced into signals normalized in this way by loss of magnitude or phase information, consider first the case of a signal consisting of two sine waves at different frequencies. For simplicity, one-dimensional signals are assumed in the analysis since the two-dimensional case follows the same reasoning. Thus the signal has values

$$\phi(t) = c_1 e^{i\omega_1 t+\theta_1} + c_2 e^{i\omega_2 t+\theta_2}$$

If the displayed signal $\hat{\phi}$ is the real part of the normalized original signal, then it has values

$$\hat{\phi}(t) \triangleq Re\,\frac{\phi(t)}{\|\phi\|} = \frac{c_1\cos(\omega_1 t+\theta_1) + c_2\cos(\omega_2 t+\theta_2)}{\sqrt{c_1^2+c_2^2}}$$

The signal ϕ_p obtained by setting the magnitudes of both terms in ϕ to a constant value $\lambda>0$ has values

$$\phi_p(t) = \lambda e^{i\omega_1 t+\theta_1} + \lambda e^{i\omega_2 t+\theta_2}$$

so that its corresponding displayed signal $\hat{\phi}$ has values

$$\hat{\phi}_p(t) \triangleq Re\,\frac{\phi_p(t)}{\|\phi_p\|} = \frac{\cos(\omega_1 t+\theta_1) + \cos(\omega_2 t+\theta_2)}{\sqrt{2}}$$

Assuming θ_1 and θ_2 are independent and uniform on $[0,2\pi]$, the mean-square distortion in the displayed signal values introduced by loss of the magnitude information is

$$E\{[\hat{\phi}(t)-\hat{\phi}_p(t)]^2\} = 1 - \frac{c_1+c_2}{\sqrt{2(c_1^2+c_2^2)}} \triangleq \delta(c_1,c_2)$$

Note that $\delta(kc_1,kc_2)=\delta(c_1,c_2)$ so that δ is constant on rays of the form $c_1=kc_2$ for $0<k<\infty$. Denoting

$$\delta(kc,c) = 1 - \frac{k+1}{\sqrt{2(k^2+1)}} \triangleq \delta(k)$$

it is apparent that $\delta(k)=0 \Leftrightarrow k=1 \Leftrightarrow c_1=c_2$. Moreover, since $\delta(k)=\delta(1/k)$, the loci of constant mean-square distortion are pairs of rays (figure 2) and the region of "small" distortion lies between these rays in the first quadrant. Also, $E\{\hat{\phi}^2(t)\}=\frac{1}{2}$, so that

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = \lim_{\alpha\to\infty} \delta(\alpha) = 1-\frac{1}{\sqrt{2}} \approx 0.3 < E\{\hat{\phi}^2(t)\}$$

In other words, the mean-square distortion in the displayed signal values introduced by setting all of the magnitudes to a positive constant λ is significantly less than the expected magnitude squared of the displayed value of the original signal -- regardless of values of c_1 and c_2.

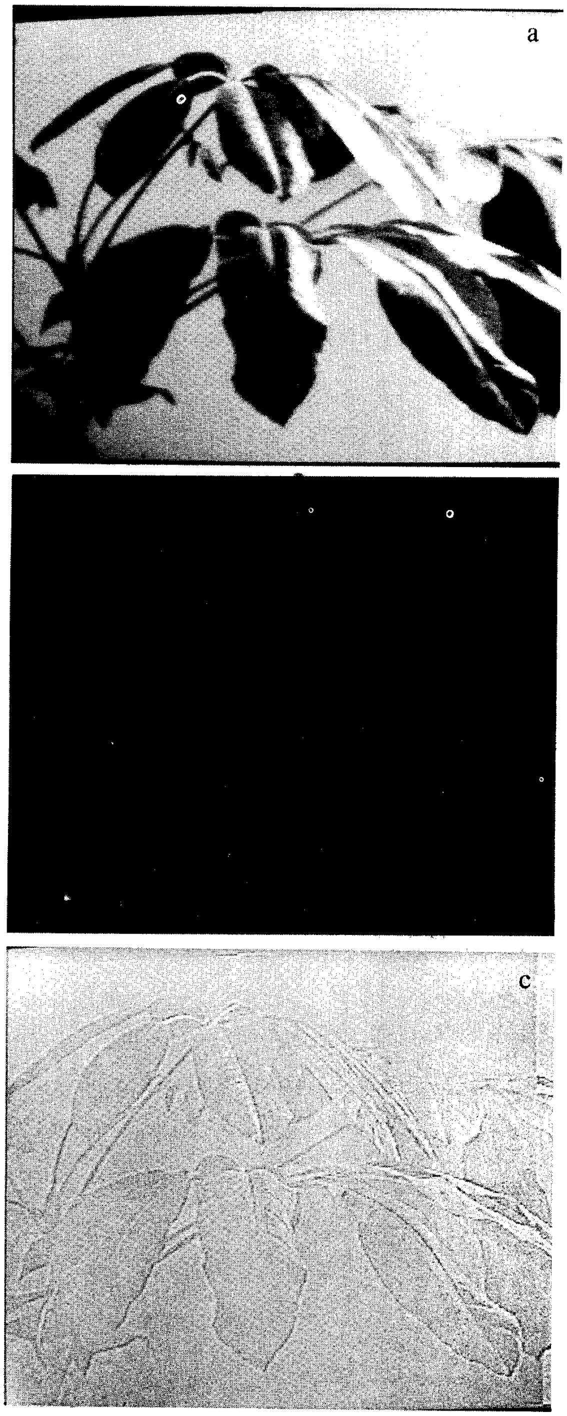

Figure 1. Normalized images: **(a)** Original $\hat{\phi}(x,y)$; **(b)** $\hat{\phi}_m(x,y)$ with magnitude information only; **(c)** $\hat{\phi}_p(x,y)$ with phase information only.

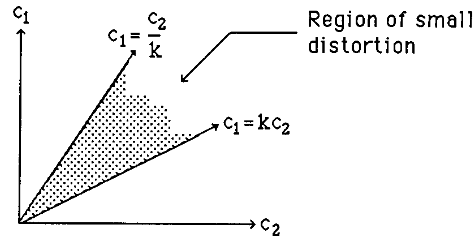

Figure 2. In the two-coefficient case, the loci of constant expected distortion are pairs of rays with reciprocal slopes. The region of "small" distortion is bounded by such a pair of rays.

3.2.2 A More General Analysis of Phase-Loss Distortion

Continuing in the vein of the above example, consider now a band-limited signal represented by a finite Fourier series

$$\phi(t) = \sum_{m=0}^{n-1} c_m e^{i(mt+\theta_m)}$$

with each c_m real and non-negative. Setting each magnitude coefficient c_m to a constant value $\lambda>0$ yields a signal with values

$$\phi_p(t) = \sum \lambda e^{i(mt+\theta_m)}$$

With normalization as described above, the displayed values of these two signals are, respectively,

$$\hat{\phi}(t) = \frac{\Sigma\, c_m \cos(mt+\theta_m)}{(\Sigma\, c_m^2)^{1/2}}$$

and

$$\hat{\phi}_p(t) = \frac{\Sigma\, \lambda \cos(mt+\theta_m)}{(\Sigma\, \lambda^2)^{1/2}} = \frac{\Sigma\, e^{i(mt+\theta_m)}}{\sqrt{n}}$$

Hence the change in the displayed values caused by elimination of the magnitude information from the signal is

$$\Delta_p(t) \doteq \hat{\phi}(t) - \hat{\phi}_p(t) = \sum_{k=0}^{n-1} \left\{ \left[\frac{c_k}{(\Sigma c_m^2)^{1/2}} - \frac{1}{\sqrt{n}} \right] \cos(kt+\theta_k) \right\}$$

Assuming the phase coefficients θ_k are independent and uniform on $[0,2\pi]$,

$$E\{\Delta_p^2\} = \frac{1}{2} \sum_k \left[\frac{c_k}{(\Sigma c_m^2)^{1/2}} - \frac{1}{\sqrt{n}} \right]^2 = 1 - \frac{\Sigma c_k}{\sqrt{n}(\Sigma c_m^2)^{1/2}}$$

The relative mean-square distortion introduced by setting the magnitude coefficients to a constant value is thus

$$D_p(t) \triangleq \frac{E\{\Delta_p^2(t)\}}{E\{\hat{\phi}^2(t)\}} = 2 - \frac{2\,\Sigma c_k}{\sqrt{n}\,(\Sigma c_m^2)^{1/2}}$$

3.2.3 Geometric Interpretation of D_p

As in the two-coefficient example above, a geometric interpretation of the region of "small" distortion is possible in this more general case. Denote by $\mathbf{P}^n$ the region in $\mathbf{R}^n$ where all coordinates are positive. The subset of $\mathbf{P}^n$ on which $D_p=\varepsilon$ is given by

$$2 - \frac{2\,\Sigma c_k}{\sqrt{n}\,(\Sigma c_m^2)^{1/2}} = \varepsilon$$

which is the equation of a right circular cone in $\mathbf{R}^n$ with vertex at the origin and axis $c_0=c_1=\cdots=c_{n-1}$. Note that the entire cone may not be contained in $\mathbf{P}^n$. In this case, the set of Fourier coefficients on which D_p is constant is the intersection of the cone with $\mathbf{P}^n$. The region $\mathbf{C}_\varepsilon$ in which $D_p \leq \varepsilon$ is the subset of $\mathbf{P}^n$ bounded by this cone, hence:

Theorem: If the Fourier coefficients c_k have joint density $\rho(\mathbf{c})$, then

$$Pr\{D_p \leq \varepsilon\} = \int_{\mathbf{C}_\varepsilon} \rho(\mathbf{c})\, d\mathbf{c}$$

3.2.4 Distortion Due to Loss of Phase

Now consider the signal with values $\phi_m(t)=\Sigma c_k e^{ikt}$ obtained from ϕ by setting all of the phase coefficients θ_k to zero. The change in the displayed values in this situation is

$$\Delta_m(t) \triangleq \hat{\phi}(t)-\hat{\phi}_m(t) = \frac{\Sigma c_k(\cos(kt+\theta_k) - \cos(kt))}{(\Sigma c_m^2)^{1/2}}$$

Again assuming the phase coefficients θ_k are independent and uniform on $[0,2\pi]$, the relative mean-square distortion caused by setting all of the phase angles to zero is

$$D_m(t) \triangleq \frac{E\{\Delta_m^2(t)\}}{E\{\hat{\phi}^2(t)\}}$$

Direct calculation yields

$$D_m(t) = \frac{\Sigma c_k^2 + 2(\Sigma c_k \cos(kt))^2}{\Sigma c_k^2} \geq 1$$

Hence the expected distortion in the displayed signal is significant regardless of the distribution of the coefficients c_k.

4. FURTHER RESULTS AND CONCLUSIONS

This note has pointed out the non-linear nature of signal normalization and has elaborated on a non-trivial interaction between signal normalization and spectrum analysis. It has been shown, in the context of a fairly general class of signals, that normalization tends to compensate for perturbations to or loss of spectral magnitude information. In this sense, spectral phase is a more critical component of such normalized signals than spectral magnitude.

Some further results in the same vein as those presented in this paper have been obtained, but are not included here for the sake of brevity. Among these are: (1) extension of the analyses presented herein to the case of multi-dimensional normalized signals; (2) consideration of normalization involving norms other than the L^2 norm; and (3) a study of distributions of spectral magnitude coefficients with respect to their effects on the sensitivity of normalized signals to perturbation or loss of magnitude or phase information.

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