Correspondence

Invariance of the Magnitude-Squared Coherence Estimate with Respect to Second-Channel Statistics

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Abstract—**Properties of the magnitude-squared coherence (MSC) estimate in the presence of spherically symmetric noise are examined from a geometric point of view. Invariance to second-channel statistics is shown and the distribution function of the estimate is derived. A relationship between spherically symmetric and Gaussian sequences is also given.**

I. Introduction

A commonly used technique for detecting the presence of a common signal in two noisy channels is to compare the magnitude-squared coherence (MSC) estimate made with sample sequences from the two channels to a threshold [1]. Thresholds corresponding to particular false alarm probabilities are typically computed using the well-known probability distribution function of the MSC estimate derived under the assumption of Gaussian processes on both channels [2]. In [3], it was shown that this distribution function is valid, regardless of the statistics of the process on one of the two channels, provided the other sequence is Gaussian.

In this correspondence, a geometric argument is presented that shows the Gaussian assumption, even on one channel, is not necessary; spherical symmetry of the distribution function of one of the sequences is sufficient to ensure invariance of the distribution of the MSC estimate with respect to the statistics of the second sequence. The distribution function of the MSC estimate is derived as a corollary to the invariance argument, and it is also shown that spherically symmetric complex sequences are Gaussian if their real and imaginary parts are independent.

Finally, invariance is shown in one important nonspherically symmetric case in which each sample in one sequence has unit magnitude and uniformly distributed phase. In this case, the distribution of the MSC estimate is identical to the solution of a classical problem in random motion.

II. Geometry of the MSC Estimate

Let $\boldsymbol{a} \triangleq \{x_n + iy_n\}_1^N$ and $\boldsymbol{b} \triangleq \{u_n + iv_n\}_1^N$ denote the complex random sequences to be used in the MSC estimate. Then the estimate is given by

$$\gamma^2(\boldsymbol{a}, \boldsymbol{b}) \triangleq \frac{|\langle \boldsymbol{a}, \boldsymbol{b}\rangle|^2}{\|\boldsymbol{a}\|^2 \cdot \|\boldsymbol{b}\|^2} = \frac{\left|\sum_{n=1}^{N} (x_n + iy_n)(u_n + iv_n)^*\right|^2}{\sum_{n=1}^{N} |x_n + iy_n|^2 \cdot \sum_{n=1}^{N} |u_n + iv_n|^2} \tag{1}$$

$$= \frac{\left\{\sum_{n=1}^{N} (x_n u_n + y_n v_n)\right\}^2 + \left\{\sum_{n=1}^{N} (-x_n v_n + y_n u_n)\right\}^2}{\sum_{n=1}^{N} (x_n^2 + y_n^2) \cdot \sum_{n=1}^{N} (u_n^2 + v_n^2)} \tag{2}$$

where * denotes complex conjugation and $\langle\ ,\ \rangle$ denotes inner product. Define unit vectors in $\boldsymbol{R}^{2N}$ by

$$\boldsymbol{\alpha} \triangleq \frac{(x_1, \cdots, x_N, y_1, \cdots, y_N)}{\sum_{n=1}^{N} \sqrt{x_n^2 + y_n^2}},$$

$$\boldsymbol{\beta}_1 \triangleq \frac{(u_1, \cdots, u_N, v_1, \cdots, v_N)}{\sum_{n=1}^{N} \sqrt{u_n^2 + v_n^2}},$$

and

$$\boldsymbol{\beta}_2 \triangleq \frac{(-v_1, \cdots, v_N, u_1, \cdots, u_N)}{\sum_{n=1}^{N} \sqrt{u_n^2 + v_n^2}}$$

so that (2) reduces to

$$\gamma^2(\boldsymbol{a}, \boldsymbol{b}) = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}_1\rangle^2 + \langle \boldsymbol{\alpha}, \boldsymbol{\beta}_2\rangle^2. \tag{3}$$

By construction, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are orthonormal. Thus, $\gamma^2(\boldsymbol{a}, \boldsymbol{b})$ is shown by (3) to be the square of the length of the projection of $\boldsymbol{\alpha}$ onto the plane defined by $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ (see Fig. 1). Hence, the distribution of γ^2 is the distribution of the square of the length of the projection of $\boldsymbol{\alpha}$ onto the plane formed by $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$.

III. Invariance with Respect to Second-Channel Statistics

In the above construction, $\boldsymbol{\alpha}$ depends only on the statistics of $\boldsymbol{a}$, and the plane of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ depends only on $\boldsymbol{b}$. If the probability density function $f_a(x_1, \cdots, x_N, y_1, \cdots, y_N)$ of $\boldsymbol{a}$ can be written as $f_a(x_1, \cdots, x_N, y_1, \cdots, y_N) = f_{ss}(\Sigma_{n=1}^N (x_n^2 + y_n^2))$ where f_{ss} is a one-dimensional function (i.e., if the value of the density depends only on the distance from the origin and not on the direction), then $\boldsymbol{a}$ is said to be *spherically symmetric*. With $\boldsymbol{a}$ spherically symmetric, then $\boldsymbol{\alpha}$ is also spherically symmetric. Furthermore, since $\boldsymbol{\alpha}$ is a vector of unit length, its spherical symmetry implies that it is uniformly distributed over the $2N$-dimensional unit sphere. If $\boldsymbol{a}$ is statistically independent of $\boldsymbol{b}$, the evaluation of the distribution of γ^2 becomes purely geometric in character, i.e., the probability that γ^2 is less than or equal to r is equal to the fraction of the $2N$-dimensional sphere that orthogonally projects into a disk of radius r centered at the origin in the plane formed by $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. Since this will not depend on the orientation of the plane (i.e., the specific occurrences of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$), we have the following.

Theorem 1: The distribution of the MSC estimate $\gamma^2(\boldsymbol{a}, \boldsymbol{b})$ does not depend on the statistics of $\boldsymbol{b}$ provided that $\boldsymbol{a}$ is spherically symmetric and statistically independent of $\boldsymbol{b}$.

Although statistical independence of $\boldsymbol{a}$ and $\boldsymbol{b}$ is assumed in the argument leading to Theorem 1, it is not a necessary condition for the result to hold. Suppose, for example, $\boldsymbol{b}$ is multiplied by the length of $\boldsymbol{a}$ to form a new sequence $\boldsymbol{b}'$. Then $\boldsymbol{b}'$ and $\boldsymbol{a}$ are not in-

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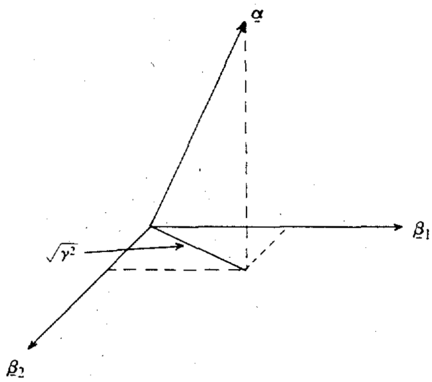

Fig. 1. The coherence estimate as a projection.

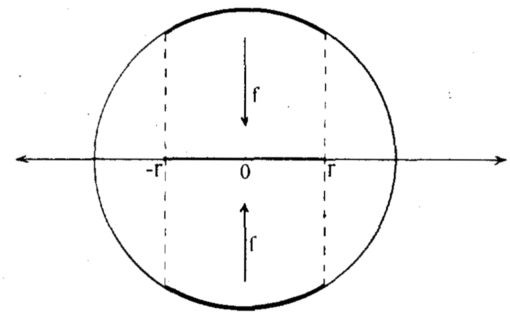

Fig. 2. For $0 \leq r \leq 1$, the probability $\Pr\{\gamma^2(\boldsymbol{a}, \boldsymbol{b}) \leq r\}$ is the fraction of the unit spherical surface in $\mathcal{R}^{2N}$ that projects into a disk of radius r.

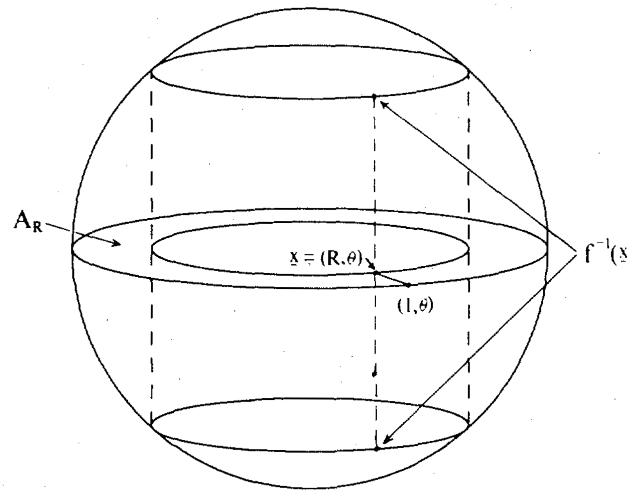

Fig. 3. The preimage of each point $(x_1, x_2) \in \boldsymbol{R}^2$ is a spherical surface in a $(2N - 2)$-dimensional space.

dependent, but $\gamma^2(\boldsymbol{a}, \boldsymbol{b}') = \gamma^2(\boldsymbol{a}, \boldsymbol{b})$, so that the distribution of the MSC estimate still does not depend on the second-channel statistics. A stronger version of Theorem 1 could be formulated with the hypothesis $f_{a|b}$ is spherically symmetric for all $\boldsymbol{b}$.

IV. Distribution of the MSC Estimate

In the previous section, the geometric character of the distribution of γ^2 under the assumptions of Theorem 1 was pointed out. In this section, the distribution of γ^2 will be derived from these geometric considerations as opposed to analytical methods that rely on the Gaussian distribution of the signals. This will be accomplished by explicitly computing the fraction of the surface of the unit sphere in $\boldsymbol{R}^{2N}$ that projects onto the annular region in $\boldsymbol{R}^2$ given in polar coordinates by $\{(r, \theta) : |R| < r \leq 1 \text{ and } 0 \leq \theta < 2\pi\}$, thereby implicitly determining the fraction of this sphere that projects into the disk of radius R centered at the origin in $\boldsymbol{R}^2$.

Consider an extension of $\{\beta_1, \beta_2\}$ to an orthonormal basis $\{\beta_1, \beta_2, \beta_3, \cdots, \beta_{2N}\}$ for $\boldsymbol{R}^{2N}$, and let $(x_1, \cdots, x_{2N})$ be coordinates in terms of this basis (see [4]). Then, assuming the hypotheses of Theorem 1, the probability that $\gamma^2(\boldsymbol{a}, \boldsymbol{b}) \leq r$ for $0 \leq r \leq 1$ is the fraction of the area of the unit spherical surface $\{\boldsymbol{x} \in \boldsymbol{R}^{2N} : x_1^2 + \cdots + x_{2N}^2 = 1\}$ that is situated over the disk $x_1^2 + x_2^2 \leq r$ (Fig. 2). To determine this area, first define the projection mapping f: $\boldsymbol{R}^{2N} \rightarrow \boldsymbol{R}^2$ by $f(x_1, \cdots, x_{2N}) = (x_1, x_2)$ and consider the preimage $f^{-1}(\bar{x}_1, \bar{x}_2)$ of a point $(\bar{x}_1, \bar{x}_2) \in \boldsymbol{R}^2$. This consists of all points $(\bar{x}_1, \bar{x}_2, x_3, \cdots, x_{2N})$ and, because $x_3, \cdots, x_{2N}$ are arbitrary, is a $(2N - 2)$-dimensional affine subspace of $\boldsymbol{R}^{2N}$ (defined in [4]).

Suppose now that $\bar{x}_1^2 + \bar{x}_2^2 = R \leq 1$. Then the intersection of $f^{-1}(\bar{x}_1, \bar{x}_2)$ with the surface of the unit sphere in $\boldsymbol{R}^{2N}$ is not empty. In fact, a point $(x_3, \cdots, x_{2N})$ in the $(2N - 2)$-dimensional space $f^{-1}(\bar{x}_1, \bar{x}_2)$ is in this intersection if and only if $x_3^2 + \cdots + x_{2N}^2 = 1 - R^2$. Thus, the points $\boldsymbol{x}$ on the surface of the unit sphere in $\boldsymbol{R}^{2N}$ for which $f(\boldsymbol{x}) = (\bar{x}_1, \bar{x}_2)$ form a spherical surface of radius $\sqrt{1 - R^2}$ in the $(2N - 2)$-dimensional space $f^{-1}(\bar{x}_1, \bar{x}_2)$.

Now, consider a line segment L_θ in $\boldsymbol{R}^2$ connecting the points (in polar coordinates) (R, θ) and $(1, \theta)$ where $R \leq 1$ (see Fig. 3). Integrating the intersection of $f^{-1}(x_1, x_2)$ and the unit spherical surface in $\boldsymbol{R}^{2N}$ over $\{(x_1, x_2) \in L_\theta\}$ yields, by the above argument, the volume (by "spherical shells") of a solid sphere of radius $\sqrt{1 - R^2}$ in a $(2N - 2)$-dimensional space [5]. This sphere has $(2N - 2)$-dimensional volume $V = \pi^{N-1}(1 - R^2)^{N-1}/(N - 1)!$. Define $A_R \triangleq \{(r, \theta) \in \boldsymbol{R}^2 : R < r \leq 1\}$. Integrating V over $0 \leq \theta < 2\pi$ yields the $(2N - 1)$-dimensional volume of the intersection of $f^{-1}(A_R)$ and the surface of the unit sphere in $\boldsymbol{R}^{2N}$. This is $2\pi V = 2\pi^N(1 - R^2)^{N-1}/(N - 1)!$. Since the $(2N - 1)$-dimensional volume of the entire surface of the unit sphere in $\boldsymbol{R}^{2N}$ is $2\pi^N/(N - 1)!$, the fraction of this spherical surface mapped by f into A_R is $(1 - R^2)^{N-1}$; the remainder of the sphere is mapped into the disk $\{(r, \theta) : 0 \leq r \leq R\}$. Hence, $\Pr\{\gamma^2(\boldsymbol{a}, \boldsymbol{b}) \leq R\} = 1 - (1 - R^2)^{N-1}$. This, of course, is the same distribution function for the MSC estimate given in [2] assuming Gaussian sequences on both channels.

V. A Comment on Spherically Symmetric Sequences

Let $\boldsymbol{a} \triangleq \{x_n + iy_n\}_1^N$ be a spherically symmetric sequence. If the real and imaginary parts of $\boldsymbol{a}$ are statistically independent, then its probability density function $f(x_1, \cdots, x_N, y_1, \cdots, y_N)$ factors as $f(x_1, \cdots, x_N, y_1, \cdots, y_N) = g(x_1, \cdots, x_N) \cdot h(y_1, \cdots, y_N)$ where g and h are the density functions of the real and imaginary parts of $\boldsymbol{a}$, respectively. Because a spherically symmetric random vector of dimension $2N$ is also spherically symmetric on any lower dimensional subspace, the real and imaginary parts of $\boldsymbol{a}$ are also spherically symmetric. Hence, $f_{ss}(\Sigma_{n=1}^N (x_n^2 + y_n^2)) = g_{ss}(\Sigma_{n=1}^N x_n^2) \cdot h_{ss}(\Sigma_{n=1}^N y_n^2)$. If f_{ss} is continuous, this relationship implies it must be Gaussian. The reason for this arises from the definition of the exponential function (see, for example, [6]). Hence, we have Theorem 2.

Theorem 2: If a sequence of complex random variables has a spherically symmetric distribution and statistically independent real and imaginary parts, then it has a Gaussian distribution.

VI. Sequences with Unit-Magnitude Samples

In some applications, the magnitudes of the elements of the two sequences may have undergone rather different distortions, and the only useful information in the sequences with respect to the presence of a common signal is contained in the phases of the sequence elements. In these circumstances, the amplitudes of the elements of both sequences are converted to a unit magnitude before the coherence estimate is evaluated. In this section, a condition is given for the distribution of the coherence estimate to be independent of second-channel statistics in this situation. Interestingly, the independence does not rely on the notion of spherical symmetry.

Assume that the samples in sequence $\boldsymbol{a}$ are statistically independent. Also, suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ have the property that each sample has unit magnitude, i.e., $x_n^2 + y_n^2 = u_n^2 + v_n^2 = 1$ for $n = 1, \cdots, N$. Then they can be written as $\boldsymbol{a} = (e^{i\theta_1}, \cdots, e^{i\theta_N})$ and $\boldsymbol{b} = (e^{i\phi_1}, \cdots, e^{i\phi_N})$. The MSC estimate becomes $\gamma^2(\boldsymbol{a}, \boldsymbol{b}) = |\Sigma_{n=1}^N \exp[i(\theta_n - \phi_n)]|^2/N^2$.

If each sample $e^{i\phi_n}$ of $\boldsymbol{a}$ is uniformly distributed on the unit circle, then so is each term $\exp[i(\theta_n - \phi_n)]$ in this expansion of $\gamma^2(\boldsymbol{a}, \boldsymbol{b})$—regardless of the distribution of the samples of $\boldsymbol{b}$. This is easily proven by convolution of an arbitrary probability density function $p(\phi_n)$ with a uniform density function, modulo 2π. Furthermore, the terms $\exp[i(\theta_n - \phi_n)]$ will be statistically independent of one another. Thus, in this case, the distribution of the MSC estimate is $1/N^2$ times the distribution of the length squared of a vector formed by adding N unit vectors in a plane which have uniformly distributed direction. This problem has a long history as the problem of a random walk in the plane. The problem of determining this distribution was posed by Pearson in 1905 (see [7]) and its history is detailed in [8].

Note that if the elements of the $\boldsymbol{a}$ sequence are allowed to have nonconstant amplitudes (i.e., $A_n = |x_n + iy_n|$) that are independently distributed, then invariance with respect to second-channel statistics is still maintained. The distribution of the coherence estimate is, in this case, related to a random walk with a random step size.

It is also important to observe that although each term of the $\boldsymbol{a}$ sequence $A_n e^{i\theta_n}$ is spherically symmetric, the $\boldsymbol{a}$ sequence itself *is not spherically symmetric*.

VII. Conclusions

We feel the important aspect of this correspondence is the demonstration of the utility of a geometric approach to this probability problem, not just for the results obtained but also for the insight into why these results come about. Although we have focused on the problem of coherence estimation with no signal present, we anticipate that the geometric approach will prove useful in the case when signal is present.

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FFT Pruning Applied to Time Domain Interpolation and Peak Localization

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Abstract—**The efficiency of the fast Fourier transform may be increased by removing operations on input values which are zero, and on output values which are not required. This is applied to interpolation of complex and real valued time domain functions. For real functions, analytic signal concepts may be used to get the Hilbert transform as a byproduct, and applied to the cross correlation function this gives an efficient and accurate method for peak localization.**

I. Introduction

Pruning of the fast Fourier transform (FFT) algorithm is the elimination of operations on zeros. When the number of input points is less than the number of transform points, the number of butterflies may be reduced. This is referred to as input pruning and was first described by Markel [1]. The efficiency and regularity of the algorithm was improved by Skinner [2] by pruning the decimation-in-time instead of the decimation-in-frequency algorithm. This algorithm requires $(N/2) \cdot \log_2 NZ$ complex multiplications where N is the transform length and NZ is the number of nonzero input values, as compared to the FFT's $(N/2) \cdot \log_2 N$ complex multiplications.

Sreenivas and Rao [3] extended the algorithm by combining input and output pruning. This applies to the situations where both the number of input points and the number of output points are less than the transform length. By viewing the decimation-in-frequency algorithm as the transpose of the decimation-in-time algorithm, Markel's pruning may be applied to the output of Skinner's algorithm, or vice versa. Furthermore, they generalized the algorithm by allowing output pruning anywhere in the output range [5]. Nagai [7] gives an alternative algorithm for generalized output pruning. It has a more regular structure than in [5], while keeping the savings in the computation. In [4] and [6], pruning algorithms are expressed using matrix formulation.

The most common application of input pruning is interpolation in the frequency domain. By appending zeros to a sequence prior to Fourier transformation, a high resolution spectrum is obtained. This is used, for instance, in autoregressive spectral analysis. The input and output pruned algorithm gives additional savings by computing the high resolution spectrum in only a preselected narrow frequency band. In [5], the amount of computation is compared to the direct DFT, unpruned FFT, and chirp Z transform methods.

In this correspondence, the application is efficient time domain interpolation. It will be shown that with some modifications, the available pruning algorithms may be applied. It will also be shown that a byproduct for real valued time samples is the Hilbert transform. It has applications in estimation of time delay found by localization of the peak of the cross correlation function.

II. Time Domain Interpolation

Consider a time sequence $x'(n)$ with N samples $(n = 0, \cdots, N - 1)$, and a new interpolating sequence $x(n)$ with rN samples, where r is a positive integer. This new sequence should have the property that $x(rn) = x'(n)$ for $n = 0, \cdots, N - 1$. This requirement distinguishes interpolation by FFT pruning from inter-

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